Implementation without Expected Utility: Ex-Post Verifiability

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Implementation without Expected Utility: Ex-Post Verifiability[1]

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Abstract

This study investigates implementation of a social choice function with complete information, where we impose various restrictions such as boundedness, permission of only small transfers, and uniqueness of iterative dominance in strict terms. We assume that the state is ex-post verifiable after the determination of allocation. We show that with three or more players, any social choice function is uniquely and exactly implementable in iterative dominance. Importantly, this study does not assume either expected utility or quasi-linearity, even if we utilize the stochastic method of mechanism design explored by Abreu and Matsushima (1992, 1994).

Keywords: Unique and Exact Implementation, Ex-post Verifiability, Non-Expected Utility, Abreu-Matsushima Mechanism.
JEL Classification Numbers: C72, D71, D78, H41

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1. Introduction

This study investigates unique and exact implementation of a social choice function under complete information. We assume that the state is *ex-post verifiable*. We design stochastic mechanisms but do *not* assume either expected utility or quasi-linearity.

The central planner attempts to achieve the allocation implied by the social choice function that is contingent on the state. The central planner, however, cannot observe the state before determining an allocation. Hence, he (or she) designs a mechanism to induce informed players to reveal their knowledges about the state. This mechanism must incentivize them to make the desirable (truthful) announcements as *unique* equilibrium behavior. The problem of this study is to clarify whether the central planner can design such effective mechanisms.[3]

We assume that the state becomes verifiable *after* the central planner determines the allocation. For instance, by conducting a follow-up survey, the central planner obtains a resultant verifiable consequence of the allocation decision that includes information about the state. The central planner utilizes this information for ex-post monetary transfers with players as a clue to detecting their lying. We show that by making ex-post monetary transfers contingent on this verified information as well as their announcements, the central planner can design a mechanism to effectively penalize any detected liar, making all players willing to make their desirable announcements.

Owing to this ex-post verifiability, we can impose the following various severe restrictions on mechanism design. We use *iterative dominance* in strict terms as the solution concept, which is defined as the set of all strategy profiles that survive through the iterative removal of strictly dominated strategies. We impose the uniqueness of such an iteratively undominated strategy for each player, even if iterative dominance is a very weak solution concept. Next, we require a mechanism to be *bounded* in the terminology of Jackson (1992): we consider only mechanisms whose message spaces are *finite*. Moreover, we use only *small* monetary transfers: we require any transfer to be close to zero off the equilibrium path, and even *no* transfers on the equilibrium path.

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[3] For surveys on implementation theory, see Moore (1992), Palfrey (1992), Osborne and Rubinstein (1994, Chapter 10), Jackson (2001), and Maskin and Sjöström (2002).

We show that any social choice function is *uniquely*, and *exactly*, implementable in iterative dominance, where we design a bounded mechanism, use only small transfers, and make no transfers on the equilibrium path. Importantly, we do *not* assume either expected utility or quasi-linearity, even if we design a *stochastic* mechanism according to the method explored by Abreu and Matsushima (1992, 1994). All we need on utility functions in this study is a much weaker condition than expected utility and quasi-linearity: *each player's utility function is continuous in monetary transfer and lottery, and is increasing in monetary transfer.*

It is well known in the implementation literature that with no ex-post verifiability, Makin-monotonicity is a necessary condition for a social choice function to be implementable in Nash equilibrium (Maskin, 1999). Maskin-monotonicity is a quite demanding condition for a deterministic social choice function. Matsushima (1988) and Abreu and Sen (1991) demonstrated a device of virtualness, which can drastically calm this difficulty by approximating a deterministic social choice function to a stochastic social choice function. However, this virtualness crucially depends on expected utility.

In contrast, this study assumes ex-post verifiability. With ex-post verifiability, Maskin-monotonicity is no longer necessary. Because of this irrelevance, we can apply the Abreu-Matsushima stochastic method even to the case without expected utility (also without quasi-linearity) with ease. The functioning of the stochastic method a la Abreu and Matsushima relies just on the *local linearity of preferences*, which is automatically implied by the above-mentioned continuity and increasingness.

There exist previous works such as Hansen (1985), Mezzetti (2004), and Deb and Mishra (2014) that incorporated verifiability into mechanism design. These works showed that verifiability makes incentive compatibility more easily satisfied. In contrast, this study's concern is the impact of verifiability on uniqueness. In this respect, this study is related to Kartik and Tercieux (2012) and Ben-Porath and Lipmann (2012), which investigated full implementation with hard evidence, stating that the great degree to which each player's showing hard evidences directly proves his (or her) announcement to be correct is crucial in full implementation. The companion paper (Matsushima, 2017) extends this study's result to the case in which the state is only partially verifiable.

The organization of this study is as follows. Section 2 shows the model that assumes continuity and increasingness, instead of expected utility and quasi-linearity. Section 3 explains small monetary transfers and ex-post verifiability. Section 4 introduces iterative dominance. Section 5 constructs a mechanism that is finite and stochastic. Section 6 shows the main theorem. Section 7 concludes.

2. The Model

We consider a situation in which the central planner determines an allocation and makes small monetary transfers. Let $N \equiv \{1,...,n\}$ denote the finite set of all players, where $n \geq 3$. Let A denote the finite set of all allocations. Let Δ denote the set of all lotteries over allocations. We denote $\alpha \in \Delta$. We will write $\alpha = a$ if $\alpha(a) = 1$. Let Ω denote the finite set of all states. A *social choice function* is defined as $f : \Omega \to \Delta$. For every $\omega \in \Omega$, the lottery over allocations $f(\omega) \in \Delta$ implies the most desirable one to achieve at the state ω. This study considers both deterministic and stochastic social choice functions.

We define the state-contingent utility function for each player $i \in N$ as

$$u_i : \Delta \times R \times \Omega \to R,$$

where $u_i(\alpha, t_i, \omega)$ implies player $i's$ utility when he (or she) expects the state ω to occur, expects the central planner to determine the allocation according to the lottery $\alpha \in \Delta$ and make a monetary transfer $t_i \in R$ to him.

We assume that $u_i(\alpha, t_i, \omega)$ is *continuous* with respect to $\alpha \in \Delta$ and $t_i \in R$, and that $u_i(\alpha, t_i, \omega)$ is *increasing* in t_i. *This study does not assume either expected utility or quasi-linearity.*

3. Small Transfers and Ex-Post Verifiability

Fix an arbitrary positive real number $\varepsilon > 0$. Because of players' limited liability, the central planner cannot make any monetary transfer that is greater than ε. We can fix ε as close to zero as possible.

The state is common knowledge among players from the beginning. However, the central planner can observe the state only after he (or she) determines the allocation. Hence, the state becomes verifiable only in the ex-post term. The central planner can make ex-post monetary transfers contingent on the state as well as the players' messages. The central planner cannot make his allocation selection contingent on the state because he must determine the allocation before he observes the state.

Based on these observations, we define a *mechanism* as $\mathrm{G} \equiv (M, g, x)$, where $M \equiv \underset{i \in N}{\times} M_i$, M_i denotes the set of all *messages* of player i, $g : M \to \Delta$ denotes the *allocation rule*, $x \equiv (x_i)_{i \in N}$ denotes the *transfer rule*, and $x_i : M \times \Omega \to [0, \varepsilon]$ denotes the transfer rule for player i. We assume that M_i is finite for all $i \in N$: we focus on mechanisms that are bounded.

Each player $i \in N$ announces a message $m_i \in M_i$ to the central planner, which is contingent on the state ω. The central planner determines the allocation according to the lottery $g(m) \in \Delta$ implied by the message profile $m \equiv (m_i)_{i \in N} \in M$ and the allocation rule g. After the state ω becomes verifiable, the central planner receives the monetary transfer $x_i(m, \omega) \in [0, \varepsilon]$ from each player i, which is implied by the message profile $m \in M$, the state $\omega \in \Omega$, and the transfer rule x_i for player i.

A *strategy for each player* i is defined as $s_i : \Omega \to M_i$, according to which player i announces the message $s_i(\omega) \in M_i$ when he observes $\omega \in \Omega$. Let S_i denote the set of all strategies for player i. Let $S \equiv \underset{i \in N}{\times} S_i$ and $s \equiv (s_i)_{i \in N} \in S$.

4. Iterative Dominance

We introduce a solution concept namely *iterative dominance*, which is defined as the survival of iterative removal of messages that are *strictly* dominated. For every $i \in N$ and $\omega \in \Omega$, let

$$M_i(0, \omega) \equiv M_i .$$

Recursively, for each $h \geq 1$, we define a subset of player $i's$ messages $M_i(h,\omega) \subset M_i$ in the manner that $m_i \in M_i(h,\omega)$ if and only if $m_i \in M_i(h-1,\omega)$, and there exists no $m_i' \in M_i(h-1,\omega)$ such that for every $m_{-i} \in M_{-i}(h-1,\omega)$,

$$u_i(g(m_i',m_{-i}),-x_i(m_i',m_{-i},\omega),\omega) > u_i(g(m),-x_i(m,\omega),\omega),$$

where we denote $M_{-i}(h-1,\omega) \equiv \underset{j \in N \setminus \{i\}}{\times} M_j(h-1,\omega)$. In this definition, in order to eliminate a message $m_i \in M_i(h-1,\omega)$, we require player i to strictly prefer another message m_i' to m_i irrespective of the other players' messages $m_{-i} \in M_{-i}(h-1,\omega)$. Let

$$M_i(\infty,\omega) \equiv \bigcap_{h=0}^{\infty} M_i(h,\omega).$$

Definition 1: A strategy $s_i \in S_i$ for player i is said to be *iteratively undominated in a mechanism* G if

$$s_i(\omega) \in M_i(\infty,\omega) \text{ for all } \omega \in \Omega.$$

Definition 2: A mechanism G is said to *uniquely implement a social choice function* f *in iterative dominance* if there exists the unique iteratively undominated strategy profile $s = (s_i)_{i \in N} \in S$ in G, i.e.,

$$M_i(\infty,\omega) = \{s_i(\omega)\} \text{ for all } \omega \in \Omega \text{ and } i \in N,$$

and it induces the value of the SCF and no monetary transfers at all times, i.e., for every $\omega \in \Omega$,

$$g(s(\omega)) = f(\omega),$$

and

$$x_i(s(\omega)) = 0 \text{ for all } i \in N.$$

A social choice function f is said to be *uniquely implementable in iterative dominance* if there exists a bounded (finite) mechanism G that uniquely implements f in iterative dominance.

Iterative dominance is a very weak solution concept. Unique implementation in iterative dominance automatically implies unique implementation in mixed strategy Nash equilibrium. This unique implementation requires not only the achievement of allocations implied by the social choice function but also zero monetary transfers on the equilibrium path. Even off the equilibrium path, only small transfers are permitted.

5. Construction

Fix an arbitrary real numbers $\eta_1 \in (0,\varepsilon)$. Let $\eta_2 \equiv \varepsilon - \eta_1 > 0$. Fix an arbitrary integer $K > 0$. Fix an arbitrary allocation $a^* \in A$. By using the stochastic method explored by Abreu and Matsushima (1992, 1994), we construct a mechanism, which is denoted by $G^* = G^*(f,\eta_1,\eta_2,K,a^*) = (M,g,x)$, in the following manner. For every $i \in N$, let

$$M_i = \underset{k=1}{\overset{K}{\times}} M_i^k,$$

and

$$M_i^k = \Omega \quad \text{for all} \quad k \in \{1,...,K\}.$$

Each player $i \in N$ announces K multiple sub-messages about the state, i.e., $m_i^k \in \Omega$ for all $k \in \{1,...,K\}$, at once, where we denote $m_i = (m_i^k)_{k=1}^K$. Let $M^k \equiv \underset{i\in N}{\times} M_i^k$ and $m^k = (m_i^k)_{i\in N} \in M^k$.

For each $k \in \{1,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for every $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } m_i^k = \omega \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \omega.$$

Note that g^k is well-defined because of $n \geq 3$. Let

$$g(m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K} \quad \text{for all} \quad m \in M.$$

The interpretation of the specified allocation rule g in G^* is as follows. The central planner randomly selects an integer k from $\{1,...,K\}$ with the same probability and determines an allocation according to the corresponding lottery $g^k(m^k) \in \Delta$. In this case, the central planner can select an allocation according to the lottery $f(\omega)$ implied by the social choice function f and the state ω whenever at least $n-1$ players i announce $m_i^k = \omega$. If there exists no such ω, he selects a^*. Since the central planner selects k at random, he selects each allocation $a \in A$ with the probability given by

$$g(m)(a) = \frac{\sum_{k=1}^{K} g^k(m^k)(a)}{K}.$$

We specify the transfer rule x_i for each player i as follows: for every $(m,\omega) \in M \times \Omega$,

$$x_i(m,\omega) = \eta_1 + \frac{r_i(m_i)}{K}\eta_2 \qquad \text{if there exists } k \in \{1,...,K\} \text{ such that}$$

$$m_i^k \neq \omega \text{, and}$$

$$m_j^{k'} = \omega \text{ for all } k' < k \text{ and } j \in N,$$

and

$$x_i(m,\omega) = \frac{r_i(m_i)}{K}\eta_2 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i(m_i) \in \{0,...,K\}$ denotes the number of the integers $k \in \{1,...,K\}$ such that $m_i^k \neq \omega$.

The interpretation of the specified transfer rule x_i for player i in G^* is as follows. If a player i is one of the *first* deviants from ω, i.e., one of the players who tell lies as the earliest sub-message among all deviants, he is fined the monetary amount η_1. Player i is also fined the monetary amount $\frac{r_i(m_i)}{K}\eta_2$ in proportion to the number of his dishonest sub-messages. (Note that announcing all his sub-messages dishonestly, he is fined the monetary amount $\eta_1 + \eta_2$ in totality.)

Note from the specifications of (η_1, η_2) that for every $i \in N$ and $(m,\omega) \in M \times \Omega$,

$$0 \le x_i(m,\omega) \le \eta_1 + \eta_2 = \varepsilon .$$

Hence, the central planner never makes monetary transfers that are greater than ε.

We denote $s_i = (s_i^k)_{k=1}^K$, where $s_i^k : \Omega \to M_i^k$. We define the *honest* strategy for player i, $s_i^* = (s_i^{*k})_{k=1}^K$, in G^* as

$$s_i^{*k}(\omega) = \omega \quad \text{for all} \quad k \in \{1,...,K\} \quad \text{and} \quad \omega \in \Omega .$$

According to s_i^*, player i makes the honest announcement for every sub-message. The honest strategy profile $s^* \equiv (s_i^*)_{i \in N}$ induces the value of the SCF f, i.e.,

$$g(s^*(\omega)) = f(\omega) \quad \text{for all} \quad \omega \in \Omega ,$$

and induces no monetary transfers, i.e.,

$$x_i(s^*(\omega), \omega) = 0 \quad \text{for all} \quad i \in N \quad \text{and} \quad \omega \in \Omega .$$

Since $u_i(\alpha, t_i, \omega)$ is continuous in (α, t_i) and increasing in t_i. we can specify K sufficiently large in the manner that whenever

(1) $$\max_{a \in A} \left| \alpha(a) - \alpha'(a) \right| \le \frac{1}{K},$$

then

(2) $$u_i(\alpha, -t_i, \omega) > u_i(\alpha', -t_i - \eta_1, \omega) \quad \text{for all} \quad t_i \in [0, \eta_2] \quad \text{and} \quad \omega \in \Omega .$$

The inequalities (2) imply that the loss from the monetary fine η_1 is greater than the gain from any change of stochastic allocation within the limit implied by (1).

6. The Theorem

The following theorem indicates that the above-specified mechanism G^* uniquely implements the social choice function f in iterative dominance. Since G^* is well-defined, we can conclude that *with ex-post verifiability and with more than two players, any social choice function is uniquely implementable in iterative dominance even if players' utility functions do not satisfy either expected utility or quasi-linearity, where we need no monetary transfers on the equilibrium path and almost no monetary transfers even off the equilibrium path.*

The Theorem: *The honest strategy profile* s^* *is the unique iteratively undominated strategy profile in* G^*.

Proof: We can show that each player $i \in N$ prefers $m_i^1 = \omega$ as follows. Suppose that there exists another player $j \in N \setminus \{i\}$ who announces $m_j^1 \neq \omega$. In this case, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined η_1 or even more, while the resultant change of allocation is within the limit implied by (1). Hence, from (2), the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation.

Next, suppose that there exists no player $j \in N \setminus \{i\}$ other than player i who announces $m_j^1 \neq \omega$. Then, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined $\frac{\eta_2}{K}$ or even more. (Note that even if he announces $m_i^1 = \omega$, he may be one of the first deviants, and therefore, he does not necessarily avoid the fine η_1.) From the specification of g and $n \geq 3$, there is no resultant change of allocation. From these observations, he prefers $m_i^1 = \omega$ regardless of the other players' announcements.

Fix an arbitrary integer $h \in \{2,...,K\}$. Suppose that any player $i \in N$ announces $m_i^{h'} = \omega$ for all $h' \in \{1,...,h-1\}$. Suppose that there exists a player $j \in N \setminus \{i\}$ other than player i who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined η_1 or even more. In the same manner as above, the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation. Next, suppose that there exists no player $j \in N \setminus \{i\}$ other than player i who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined $\frac{\eta_2}{K}$ or even more. In the same manner as above, there is no resultant change of allocation in this case. Hence, he prefers $m_i^h = \omega$.

Q.E.D.

The proof of the theorem is similar to the proofs of the main theorems in Abreu and Matsushima (1992, 1994). However, there is a substantial difference between these works and this study: Abreu and Matsushima did not use ex-post verifiability. It is well known in the implementation literature that without ex-post verifiability, Maskin-monotonicity is a necessary condition for a social choice function to be implementable in Nash equilibrium. Maskin-monotonicity is a quite demanding requirement if we consider deterministic social choice functions.

Matsushima (1988) and Abreu and Sen (1991) showed that a deterministic social choice function fails to satisfy Maskin-monotonicity, but there always exists a stochastic social choice function that is virtually the same as this deterministic social choice function and satisfies Maskin-monotonicity. However, this virtualness relies crucially on expected utility: we cannot directly extend Abreu and Matsushima (1992, 1994) to the case without expected utility.

In contrast, this study assumes ex-post verifiability. With ex-post verifiability, Maskin-monotonicity is no longer a necessary condition. Abreu and Matsushima (1992, 1994) developed a stochastic method of iteratively eliminating unwanted equilibria. The main theorem of this study shows that this stochastic method functions even without expected utility: all we need to apply this method to the case without expected utility is just to assume that $u_i(\alpha, t_i, \omega)$ is continuous in (α, t_i) and increasing in t_i.

10. Conclusion

We investigated unique and exact implementation of a social choice function under complete information, where we required a mechanism to be bounded, utilize only small monetary transfers, and satisfy uniqueness of iterative dominance. By assuming that the state is ex-post verifiable, we showed that any social choice function is uniquely and exactly implementable in iterative dominance. This permissive result does not assume either expected utility or quasi-linearity, even if stochastic mechanisms are used. This study is the first work to analyze bounded mechanism design with uniqueness of mixed strategy Nash equilibrium without expected utility.

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