

**Bootstrap Confidence Bands for Special
Estimation of Lévy Densities under
High-Frequency Observation**

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BOOTSTRAP CONFIDENCE BANDS FOR SPECTRAL ESTIMATION OF LÉVY DENSITIES UNDER HIGH-FREQUENCY OBSERVATIONS

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ABSTRACT. This paper develops bootstrap methods to construct uniform confidence bands for nonparametric spectral estimation of Lévy densities under high-frequency observations. We assume that we observe n discrete observations at frequency $1/\Delta > 0$, and work with the high-frequency setup where $\Delta = \Delta_n \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$. We employ a spectral (or Fourier-based) estimator of the Lévy density, and develop novel implementations of Gaussian multiplier (or wild) and empirical (or Efron's) bootstraps to construct confidence bands for the spectral estimator on a compact set that does not intersect the origin. We provide conditions under which the proposed confidence bands are asymptotically valid. Our confidence bands are shown to be asymptotically valid for a wide class of Lévy processes. We also develop a practical method for bandwidth selection, and conduct simulation studies to investigate the finite sample performance of the proposed confidence bands.

1. INTRODUCTION

In the financial economics literature, it has been argued that the presence of jumps plays an important role in the dynamics of financial data such as asset returns, interest rates, currencies, and so on [cf. 25, 1, 41, 3]. A *Lévy process* is a fundamental class of continuous-time stochastic processes allowing for jumps; we refer to [58] and [10] as standard references on Lévy processes. From the Lévy-Itô decomposition [58, Theorem 19.2], a Lévy process is decomposed into the sum of drift, Brownian, and jump components, and the distribution of the Lévy process is completely determined by the three parameters, namely, the drift, the diffusion coefficient, and the *Lévy measure*. The Lévy measure controls the jump dynamics of the Lévy process, and therefore, inference on the Lévy measure is of particular interest. In this paper, we assume that the Lévy density has a Lebesgue density (Lévy density), and study inference on the Lévy density from *high-frequency* observations. High-frequency data – data collected for every minute, second, or even microsecond – have become available due to the advancement of information technologies, and the analysis of high-frequency financial data has attracted a great deal of attentions in the financial econometrics literature; see, e.g., [3].

To be precise, we work with the following setting. Let $L = (L_t)_{t \geq 0}$ be a Lévy process, i.e., L is a stochastic process starting at 0 with stationary independent increments and càdlàg sample paths. From the Lévy-Khinchin representation [58, Theorem 8.1], L_t has characteristic function $\varphi_t(u) = \mathbb{E}[e^{iuL_t}] = e^{t\psi(u)}$, $u \in \mathbb{R}$, where $i = \sqrt{-1}$, and

$$\psi(u) = -\frac{u^2\sigma^2}{2} + iu\gamma + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{[-1,1]}(x))\nu(dx).$$

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The triplet (σ^2, γ, ν) , called the Lévy triplet, completely characterizes the distribution of the Lévy process L [cf. 58, Theorem 7.10]. Specifically, $\sigma^2 \geq 0$ is the diffusion coefficient, $\gamma \in \mathbb{R}$ is the drift, and ν is the Lévy measure, i.e., a Borel measure on \mathbb{R} such that

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty \quad \text{and} \quad \nu(\{0\}) = 0.$$

For any (Borel) set $A \subset \mathbb{R}$, $\nu(A)$ coincides with the expected number of jumps falling in $A \setminus \{0\}$ in the unit time: $\nu(A) = \mathbb{E}[\sum_{0 < t \leq 1} 1(L_t - L_{t-} \in A \setminus \{0\})]$, where $L_{t-} = \lim_{s \uparrow t} L_s$ (recall that L has at most countably many jumps on $(0, t]$ for any $t > 0$). In this paper, we assume that the Lévy measure has Lebesgue density ρ , called the Lévy density, i.e., $\nu(dx) = \rho(x)dx$. Furthermore, we assume that we observe discrete observations $L_{j\Delta}$, $j = 1, \dots, n$ at frequency $1/\Delta > 0$, and work with the high-frequency setup where $\Delta = \Delta_n \rightarrow 0$ and $n\Delta \rightarrow \infty$ as $n \rightarrow \infty$. Since we are interested in estimation of the Lévy measure (or more precisely its Lebesgue density), we require $n\Delta \rightarrow \infty$. Heuristically, this can be understood from the observation that, within any fixed time interval, say the unit time, the Lévy process $(L_t)_{t \in [0,1]}$ has only finitely many jumps that fall in a local neighborhood not containing the origin, so that even if the whole path $(L_t)_{t \in [0,1]}$ could be observed, there are only finitely many data that can be used to estimate the Lévy measure at the local neighborhood. Concretely, we have in mind that the unit time is one day, and if we have 6.5 trading hours in a day and take 5 minutes as a time span, then $\Delta = 5/(6.5 \times 60) = 1/78 \approx 0.013$; each year has around 252 business days, and so we have $78 \times 252 = 19656$ observations per a year.

Under this setup, the goal of this paper is to develop bootstrap methods to construct confidence bands for the Lévy density. Since the Lévy density can blow up around the origin, we focus on confidence bands on a compact set that does not intersect the origin. We employ a spectral (or Fourier-based) estimator of the Lévy density, and develop novel implementations of Gaussian multiplier and empirical bootstraps to construct confidence bands for the spectral estimator. We provide conditions under which the proposed confidence bands are asymptotically valid. Notably, our confidence bands are shown to be asymptotically valid for a wide class of Lévy processes, including compound Poisson processes, (Variance-) Gamma processes, Inverse Gaussian processes, tempered stable processes, and Normal Inverse Gaussian processes with or without Brownian components.¹ Confidence bands provide a simple graphical description of the accuracy of a nonparametric curve estimator, thereby quantifying uncertainties of the estimator simultaneously over (in most cases continuum of) designs points, which is of practical importance in statistical analysis. Despite extensive studies on consistent estimation of the Lévy density, however, research on confidence intervals or bands for the Lévy density is relatively scarce – see a literature review below. In particular, to the best of our knowledge, this is the first paper to derive bootstrap-based confidence bands for the Lévy density. In addition to the theoretical results, we also develop a practical method for bandwidth selection, inspired by [11], and conduct simulation studies to investigate the finite sample performance of the proposed confidence bands.

The literature on nonparametric estimation of Lévy measures or densities is broad. Recent contributions include [59, 29, 21, 22, 24, 44, 26, 7] under the high-frequency setup (i.e., $\Delta = \Delta_n \rightarrow 0$ as $n \rightarrow \infty$), and [27, 54, 35, 14, 22, 44, 8, 36, 43, 62, 9] under the low-frequency setup (i.e., $\Delta > 0$ is fixed). [42] study nonparametric estimation of the Lévy measure for a

¹For tempered stable processes, however, the stability index has is at most 1 for a technical reason.

Lévy driven Ornstein-Uhlenbeck process under high and low frequency observations. [55, 56] study estimation of distribution functions such as $\int_{-\infty}^{\cdot} (1 \wedge x^2) \nu(dx)$, and prove Donsker-type functional limit theorems for distributional function estimators under low- and high-frequency setups, respectively. We also refer to [13, 64, 12, 38] for inference on Lévy measures. However, none of these papers studies confidence bands for Lévy densities.

To the best of our knowledge, [31] and its follow-up paper [48] are the only references that derive uniform confidence bands for Lévy densities. They work with the high-frequency setup, but employ sieve (or projection) estimators based on the observation that $\Delta^{-1}P(L_{\Delta} \geq x) \approx \nu([x, \infty))$ for $x > 0$ [see also 29], which are substantially different from our spectral estimator. So, first of all, their results do not cover ours. Similarly to [60, 6], [31] proves that the supremum deviation of the sieve estimator, suitably normalized, converges in distribution to a Gumbel distribution, by using the Komlós-Major-Tusnády (KMT) approximation of the empirical distribution function by Brownian bridges [47], combined with extreme value theory. [31] uses the Gumbel approximation to construct analytic confidence bands for the Lévy density, but does not study bootstrap-based confidence bands. However, since the convergence of normal extremes is known to be slow [37], in standard nonparametric density and regression function estimation, it is often recommended to use versions of bootstraps to construct confidence bands, instead of relying on Gumbel approximations [cf. 53, 20, 15].² This paper contributes to the literature on nonparametric inference for Lévy processes by developing bootstrap confidence bands for the first time in the Lévy density estimation case. Furthermore, spectral-type estimators are among the most commonly used methods for estimation of the Lévy density [cf. 24, 9], and developing inference methods for them is practically important.

From a technical point of view, the proofs of the main theorems build on non-trivial applications of the intermediate Gaussian and bootstrap approximation theorems developed in [15, 16, 18]. The analysis of the present paper has some connections to those of [45, 46] that study confidence bands for deconvolution and nonparametric errors-in-variables regression, respectively. However, the high-frequency setup in Lévy process estimation has different probabilistic structures than the i.i.d. setup in standard nonparametric estimation. For example, the increment distribution P_{Δ} (i.e., the distribution of L_{Δ}) need not be continuous and may have a discrete component (which is in contrast to the standard density estimation case); P_{Δ} is indexed by Δ with $\Delta = \Delta_n \rightarrow 0$ as $n \rightarrow \infty$, and degenerates to the point mass at the origin; and the interplay between Δ and n has to be taken care of. In particular, providing low-level regularity conditions for validity of bootstrap confidence bands in the Lévy density estimation case is far from trivial and requires substantial work. See the discussion after Assumption 4.1 and Section 4.2.

In this paper, we assume that data do not contain microstructure noises. We have in mind that the time span Δ is small but not too small – say 5 minutes if the unit time is one day. For such cases, [4] argue that the effect of microstructure noise is small.

The rest of the paper is organized as follows. In Section 2, we define a spectral estimator for the Lévy density, and in Section 3 we describe our bootstrap methods to construct confidence bands for the spectral estimator. We consider two bootstrap methods, namely, Gaussian multiplier and empirical bootstraps. In Section 4, we present theorems that establish asymptotic validity of the proposed confidence bands. In Section 5, we provide concrete examples of Lévy processes that

²For the trigonometric basis, [48] develop an analytical method based on higher order expansions to improve on the Gumbel approximation; see their Theorem 3.7.

satisfy our regularity conditions. In Section 6, we propose a practical method for bandwidth selection, and study its finite sample performance via numerical simulations. All the proofs are deferred to Sections 7 and 8.

1.1. Notation. For any non-empty set T and any (complex-valued) function f on T , let $\|f\|_T = \sup_{t \in T} |f(t)|$. Let $\ell^\infty(T)$ denote the (real) Banach space of all bounded real-valued functions on T equipped with the sup-norm $\|\cdot\|_T$. The Fourier transform of an integrable function f on \mathbb{R} is defined as $\varphi_f(u) = \int_{\mathbb{R}} e^{iux} f(x) dx$ for $u \in \mathbb{R}$. For any $x \in \mathbb{R}$, let δ_x denote the Dirac measure at x . For any $a, b \in \mathbb{R}$, let $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For $a \in \mathbb{R}$ and $b > 0$, we use the shorthand notation $[a \pm b] = [a - b, a + b]$. For any non-empty set A in \mathbb{R} and any $\varepsilon > 0$, let $A^\varepsilon = \{x \in \mathbb{R} : d(x, A) \leq \varepsilon\}$ where $d(x, A) = \inf_{y \in A} |x - y|$. For any positive sequences a_n, b_n , we write $a_n \lesssim b_n$ if there is a positive constant $C > 0$ independent of n such that $a_n \leq Cb_n$ for all n , $a_n \sim b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$, and $a_n \ll b_n$ if $a_n/b_n \rightarrow 0$ as $n \rightarrow \infty$.

2. SPECTRAL ESTIMATION OF LÉVY DENSITY

We first describe a spectral estimation method for Lévy densities. Let $Y_{n,j} = L_{j\Delta} - L_{(j-1)\Delta}$, $j = 1, \dots, n$ be increments of discrete observations of the Lévy process L , and observe that $Y_{n,j}$, $j = 1, \dots, n$ are i.i.d. whose common characteristic function is $\varphi_\Delta(u) = \mathbb{E}[e^{iuL\Delta}] = e^{\Delta\psi(u)}$. In this paper, we assume that

$$\int_{\mathbb{R}} x^2 \rho(x) dx < \infty, \quad (2.1)$$

which is equivalent to assuming that $\mathbb{E}[L_1^2] < \infty$ (and $\mathbb{E}[L_t^2] < \infty$ for all $t > 0$; see [58], Corollary 25.8). Condition (2.1) ensures that the integral $\int_{\mathbb{R}} (e^{iux} - 1 - iux)\rho(x) dx$ is well-defined, so that the characteristic exponent $\psi(u)$ admits an alternative representation:

$$\psi(u) = -\frac{u^2 \sigma^2}{2} + iu\gamma_c + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\rho(x) dx,$$

where $\gamma_c = i^{-1}(e^{\psi(u)})'|_{u=0} = \mathbb{E}[L_1]$. Furthermore, under Condition (2.1), differentiating $\psi(u)$ twice, we arrive at the key identity

$$\psi''(u) = \frac{\varphi''_\Delta(u)\varphi_\Delta(u) - (\varphi'_\Delta(u))^2}{\Delta\varphi_\Delta^2(u)} = -\sigma^2 - \int_{\mathbb{R}} e^{iux} x^2 \rho(x) dx.$$

Therefore, applying the Fourier inversion, we conclude that

$$\begin{aligned} x^2 \rho(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (-\psi''(u) - \sigma^2) du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \left(\frac{(\varphi'_\Delta(u))^2 - \varphi''_\Delta(u)\varphi_\Delta(u)}{\Delta\varphi_\Delta^2(u)} - \sigma^2 \right) du, \end{aligned} \quad (2.2)$$

where the Fourier inversion should be interpreted in the distributional sense if the integral is not well-defined. This expression leads to a method to estimate ρ .

First, we estimate $\varphi_\Delta^{(k)}(u)$, $k = 0, 1, 2$ by $\widehat{\varphi}_\Delta^{(k)}(u)$, where

$$\widehat{\varphi}_\Delta(u) = \frac{1}{n} \sum_{j=1}^n e^{iuY_{n,j}}, \quad u \in \mathbb{R}$$

is the empirical characteristic function ($\varphi_\Delta^{(k)}$ denotes the k -th derivative of φ_Δ with $\varphi_\Delta^{(0)} = \varphi_\Delta$). Let $W : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function (kernel) such that $\int_{\mathbb{R}} W(x)dx = 1$ and its Fourier transform φ_W is supported in $[-1, 1]$ (i.e., $\varphi_W(u) = 0$ for all $|u| > 1$). Then the spectral estimator of ρ is defined by

$$\hat{\rho}(x) = \frac{1}{2\pi x^2} \int_{\mathbb{R}} e^{-iux} \left(\frac{(\hat{\varphi}'_\Delta(u))^2 - \hat{\varphi}''_\Delta(u)\hat{\varphi}_\Delta(u)}{\Delta \hat{\varphi}_\Delta^2(u)} - \hat{\sigma}^2 \right) \varphi_W(uh) du \quad (2.3)$$

for $x \neq 0$, where $h = h_n$ is a sequence of positive numbers (bandwidths) such that $h_n \rightarrow 0$ as $n \rightarrow \infty$, and $\hat{\sigma}^2$ is a pilot estimator of σ^2 .

Some comments on the spectral estimator $\hat{\rho}$ are in order. First, as long as $h \gtrsim \Delta^{1/2}$, $\inf_{|u| \leq h^{-1}} |\hat{\varphi}_\Delta(u)| \gtrsim 1 - o_{\mathbb{P}}(1)$, so that $\hat{\rho}(x)$ for $x \neq 0$ is well-defined with probability approaching one (see Lemmas 7.2 and 7.4). Second, the function $\hat{\rho}$ is real-valued. Third, noting that $\hat{\psi} := \Delta^{-1} \log \hat{\varphi}_\Delta$ is well-defined on $[-h^{-1}, h^{-1}]$ (with probability approaching one) as the *distinguished logarithm* [19, Theorem 7.6.2], we see that the spectral estimator $\hat{\rho}$ can be alternatively expressed as

$$\hat{\rho}(x) = \frac{1}{2\pi x^2} \int_{\mathbb{R}} e^{-iux} \left(-\hat{\psi}''(u) - \hat{\sigma}^2 \right) \varphi_W(uh) du$$

for $x \neq 0$. Finally, in this paper, we are interested in estimating ρ on a compact set I away from the origin (e.g. $[a, b] \cup [c, d]$ for $a < b < 0 < c < d$), and therefore, as long as $|W(x)|$ decays sufficiently fast as $|x| \rightarrow \infty$, we may take $\hat{\sigma}^2 = 0$ in theory. See the discussion after Assumption 4.1 below. However, in our simulation studies, we found that, in case of $\sigma > 0$, using a proper estimator for $\hat{\sigma}^2$ improves on the empirical performance of the estimator $\hat{\rho}$ and the inference methods, especially if the set I is close to the origin. Therefore, we recommend to plug-in a proper estimator of σ^2 . There are several existing estimators for σ^2 ; see Example 2.1 below.

Our spectral estimator (2.3) is considered and studied in [8] and [36] under the low-frequency setup. [56] use the spectral estimator (2.3) to construct estimators for distribution functions such as $\int_{-\infty}^{\cdot} (1 \wedge x^2) \nu(dx)$, and prove Donsker-type functional limit theorems for distribution function estimators under the high-frequency setup. There are versions of spectral-type estimators of ρ similar to but different from ours (2.3). For example, in case of $\sigma = 0$, [23] consider simplified versions of the estimator (2.3) by replacing $\hat{\varphi}_\Delta(u)$ with 1 and/or $\hat{\varphi}'_\Delta(u)$ with 0. Such simplifications produce additional biases that depend on Δ (but not on smoothness of $x^2\rho$); since the problem of bias is already delicate in construction of confidence bands in standard nonparametric estimation [cf. 65, Section 5.7], producing additional biases is not favorable to our goal from both theoretical and practical view points. Hence, in this paper, we focus on the current spectral estimator (2.3). It is worth pointing out that the identification (2.2) of the Lévy density ρ holds without relying on the assumption that $\Delta \rightarrow 0$, and therefore the deterministic bias of our spectral estimator $\hat{\rho}$ does not depend on Δ ; see the discussion after Assumption 4.1 below.

Furthermore, under relatively mild conditions, our spectral estimator (2.3) is consistent under the weighted sup-norm on \mathbb{R} , $\|f\|_{w, \infty} = \sup_{x \in \mathbb{R}} |x^2 f(x)|$ (see Section 9), and thereby is able to capture the shape of ρ *globally* (i.e., uniformly over $\mathbb{R} \setminus \{0\}$), which we believe is an attractive feature of the spectral estimator $\hat{\rho}$.

Example 2.1 (Examples of estimators for σ^2). There are several consistent estimators for σ^2 available in the literature on high-frequency data analysis for continuous-time stochastic processes. We provide a couple of examples here. The first example is the truncated realized

volatility (TRV) estimator proposed in [50] :

$$\widehat{\sigma}_{TRV}^2 = \frac{1}{n\Delta} \sum_{j=1}^n Y_{n,j}^2 1_{\{|Y_{n,j}| \leq \alpha_0 \Delta^{\theta_0}\}}, \quad (2.4)$$

where $\alpha_0 > 0$ and $\theta_0 \in (0, 1/2)$. The second example is the power variation (PV) estimator proposed in [5]:

$$\widehat{\sigma}_{PV}^2(\alpha) = \left(\frac{1}{n\Delta^{\alpha/2} m_\alpha} \sum_{j=1}^n |Y_{n,j}|^\alpha \right)^{2/\alpha},$$

where $\alpha \in (0, 2)$ and $m_\alpha = 2^{\alpha/2} \Gamma((\alpha + 1)/2) / \sqrt{\pi}$ is the α -th absolute moment of $N(0, 1)$. [40] study the optimal rate of convergence for estimating σ^2 in the minimax sense and propose the following estimator (modified to our setup):

$$\widehat{\sigma}_{JR}^2 = -\frac{2}{\Delta u_n^2} (\log |\widehat{\varphi}_\Delta(u_n)|) 1_{\{\widehat{\varphi}_\Delta(u_n) \neq 0\}},$$

where $u_n \propto \sqrt{(\log n)/\Delta}$ is a deterministic sequence. In our simulation studies, we use $\widehat{\sigma}_{TRV}^2$ as an estimator of σ^2 .

Strictly speaking, the references cited above study the asymptotic properties of the estimators under a different high-frequency setup that $\Delta \rightarrow 0$ as $n \rightarrow \infty$ but $n\Delta$ is fixed. For the asymptotic properties of the PV and JR estimators under our setup, we refer to Comte and Genon-Catalot [23, Proposition 5.3] [see also 2] and Nickl et al. [56, Proposition 13], respectively. For the sake of completeness, we summarize the asymptotic properties of the TRV estimator in the following lemma. See Section 8.1 for the proof.

Lemma 2.1. *Suppose that the Lévy measure satisfies $\int_{[-1,1]} |x|^\alpha \nu(dx) < \infty$ for some $\alpha \in (0, 2)$, and if $\alpha \in [1, 2)$, then assume in addition that $\int_{[-1,1]^c} |x| \nu(dx) < \infty$. Then $|\widehat{\sigma}_{TRV}^2 - \sigma^2| = O_{\mathbb{P}}(n^{-1/2} + \Delta^{(2-\alpha)\theta_0})$.*

2.1. Comparison with direct kernel estimation: preliminary simulations. Alternatively to spectral-type estimation methods, exploiting the assumption that $\Delta \rightarrow 0$ as $n \rightarrow \infty$, we can estimate the Lévy density $\rho(x)$ for $x \neq 0$ by applying directly kernel density estimation to increments $Y_{n,1}, \dots, Y_{n,n}$, i.e.,

$$\widehat{\rho}^{direct}(x) = \frac{1}{n\Delta h} \sum_{j=1}^n W^{direct}((x - Y_{n,j})/h), \quad (2.5)$$

where $W^{direct} : \mathbb{R} \rightarrow \mathbb{R}$ is a compactly supported smooth kernel function. In fact, for a given fixed $x \neq 0$, if ρ is Lipschitz continuous in a neighborhood of x , then from Lemma B.1 and Proposition 2.1 in [31], we can show that

$$\mathbb{E}[\widehat{\rho}^{direct}(x)] = \frac{1}{h} \underbrace{\int_{\mathbb{R}} W^{direct}((x - y)/h) \rho(y) dy}_{=\rho(x) + O(h)} + O(\Delta/h),$$

so that $\widehat{\rho}^{direct}$ is consistent for ρ at x under appropriate regularity conditions. Actually, the direct kernel estimator (2.5) is mentioned in Figueroa-López [30, Section 4.1], although the detailed properties of (2.5) are not studied there. From a theoretical point of view, it is rather easier to

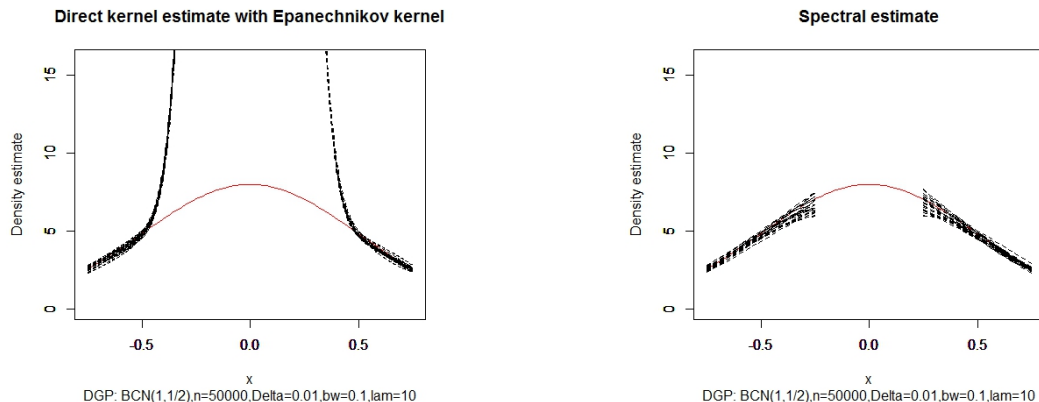


FIGURE 1. Plots of 25 realizations of direct kernel estimates with Epanechnikov kernel (left) and spectral estimates with a flap-top kernel (right) on $[-0.75, -0.25] \cup [0.25, 0.75]$ for $L_t = B_t + J_t$ where B_t is a standard Brownian motion and J_t is a compound Poisson process with intensity $\lambda = 10$ and jump size distribution $N(0, v^2)$ ($v = 0.5$). The bandwidth value is $h = \sqrt{\Delta} = 0.1$. The red lines correspond to the true Lévy density $\rho(x) = \lambda e^{-x^2/(2v^2)}/\sqrt{2\pi v^2}$.

develop inference methods for $\hat{\rho}^{direct}$ than the spectral estimator (2.3) under the high-frequency setup, since the former is of simpler form than the latter. So, one might be tempted to wonder why we bother to use a more complicated estimator $\hat{\rho}$.

It turns out that, however, in the finite sample, the direct kernel estimate (2.5) tends to have (much) larger biases, especially near the origin, than the spectral estimate (2.3). Figures 1 and 2 compare realizations of direct kernel estimates with Epanechnikov kernel and spectral estimates with a flap-top kernel for a jump-diffusion process $L_t = B_t + J_t$ where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion and $J = (J_t)_{t \geq 0}$ is a compound Poisson process (independent of B) with intensity $\lambda = 10$ and jump size distribution $N(0, v^2)$ ($v = 0.5$), and for a Gamma process with parameter $(c_+, \lambda_+) = (0.2, 1)$ (i.e., L_Δ has Gamma distribution with shape parameter $c_+ \Delta$ and scale parameter 1). Here we set $(n, \Delta) = (50000, 0.01)$. Preliminary calibrations show that $h = \sqrt{\Delta} = 0.1$ works well for for both estimates, and we set this bandwidth value to generate these figures. The flap-top kernel used for the spectral estimate is defined by the inverse Fourier transform of equation (4.4) ahead with $b = 1$ and $c = 0.05$. Furthermore, for the spectral estimate, we plug-in the TRV estimate $\hat{\sigma}_{TRV}^2$ with $\alpha_0 = 3$ and $\theta_0 = 0.48$ for the jump-diffusion case, and set $\hat{\sigma}^2 = 0$ for the Gamma process case (using the TRV estimate for the Gamma process case produced almost same simulation results).

These figures show that the direct kernel estimate has large biases especially near the origin, whereas the spectral estimate performs reasonably well on the entire set for each case ($[-0.75, -0.25] \cup [0.25, 0.75]$ for the jump-diffusion case, and $[0.25, 0.75]$ for the Gamma process case). In particular, the direct kernel estimate exhibits unreasonable behaviors for $|x| < 0.5$ in the jump-diffusion case. Intuitively, such unreasonable behaviors can be understood from the observation that the increment distribution P_Δ (i.e., the distribution of L_Δ) of the jump-diffusion process is the convolution of $N(0, \Delta v^2)$ with the compound Poisson distribution that has a point mass at the origin, and therefore, the density of P_Δ has a sharp peak around the origin. Since

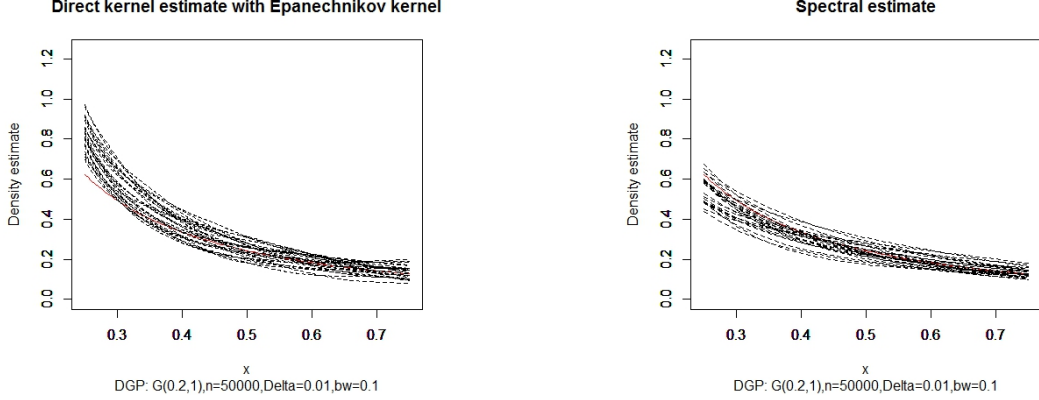


FIGURE 2. Plots of 25 realizations of direct kernel estimates with Epanechnikov kernel (left) and spectral estimates with a flap-top kernel (right) on $[0.25, 0.75]$ where L_t is a Gamma process with parameter $(c_+, \lambda_+) = (0.2, 1)$, i.e., L_Δ has Gamma distribution with shape parameter $c_+\Delta$ and scale parameter λ_+ . The bandwidth value is $h = \sqrt{\Delta} = 0.1$. The red lines correspond to the true Lévy density $\rho(x) = c_+x^{-1}e^{-\lambda_+x}1_{(0,\infty)}(x)$.

the direct kernel estimate is estimating the density of P_Δ scaled by Δ^{-1} , it tends to return unreasonably large values near the origin (we also note that, since $P(|N(0, v^2)| \geq 0.5) \approx 0.3$ for $v = 0.5$, the interval $(-0.5, 0.5)$ is not a “small” neighborhood of the origin). These preliminary simulation results motivate us to study inference methods for the spectral estimator (2.3) rather than the direct kernel estimator (2.5).

3. CONSTRUCTION OF CONFIDENCE BANDS

We consider to construct confidence bands for ρ on a compact set I in $\mathbb{R} \setminus \{0\}$. For example, $I = [a, b]$ for $0 < a < b$ or $a < b < 0$; or $I = [a, b] \cup [c, d]$ for $a < b < 0 < c < d$. The set I may be a singleton, i.e., $I = \{x_0\}$ for $x_0 \neq 0$, although we are primarily interested in the above two cases.

Under the regularity conditions stated below, we will show that $x^2(\hat{\rho}(x) - \rho(x))$ can be approximated by

$$\frac{-1}{2\pi\Delta} \int_{\mathbb{R}} e^{-iux} \frac{(\hat{\varphi}_\Delta - \varphi_\Delta)''}{\varphi_\Delta}(u) \varphi_W(uh) du \quad (3.1)$$

uniformly in $x \in I$. By a change of variables, we may rewrite the term (3.1) as

$$\frac{1}{n\Delta h} \sum_{j=1}^n \{Y_{n,j}^2 K_n((x - Y_{n,j})/h) - E[Y_{n,1}^2 K_n((x - Y_{n,1})/h)]\}, \quad (3.2)$$

where K_n is the function defined by

$$K_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{\varphi_W(u)}{\varphi_\Delta(u/h)} du, \quad x \in \mathbb{R}.$$

Note that K_n is well-defined and real-valued. Define

$$s_n^2(x) = \text{Var}(Y_{n,1}^2 K_n((x - Y_{n,1})/h)), \quad s_n(x) = \sqrt{s_n^2(x)}, \quad x \in I,$$

and consider the process

$$Z_n(x) = \frac{1}{s_n(x)\sqrt{n}} \sum_{j=1}^n \{Y_{n,j}^2 K_n((x - Y_{n,j})/h) - \mathbb{E}[Y_{n,1}^2 K_n((x - Y_{n,1})/h)]\}, \quad x \in I.$$

Under the regularity conditions stated below, $\inf_{x \in I} s_n^2(x) \gtrsim \Delta h$ for sufficiently large n . Furthermore, we will show that there exists a tight Gaussian random variable Z_n^G in $\ell^\infty(I)$ with mean zero and the same covariance function as Z_n , and such that the distribution of $\|Z_n^G\|_I = \sup_{x \in I} |Z_n^G(x)|$ asymptotically approximates that of $\|Z_n\|_I$ in the sense that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|Z_n\|_I \leq z\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \rightarrow 0$$

as $n \rightarrow \infty$, which in turn yields that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|\sqrt{n}\Delta h x^2(\hat{\rho} - \rho)/s_n\|_I \leq z\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \rightarrow 0. \quad (3.3)$$

Hence, construction of confidence bands for ρ on I reduces to approximating or estimating quantiles of $\|Z_n^G\|_I$. In fact, let

$$\begin{aligned} c_n^G(1 - \tau) &= (1 - \tau)\text{-quantile of } \|Z_n^G\|_I \\ &:= \inf\{z \in \mathbb{R} : \mathbb{P}\{\|Z_n^G\|_I \leq z\} \geq 1 - \tau \end{aligned}$$

for $\tau \in (0, 1)$, and consider a confidence band of the form

$$\widehat{\mathcal{C}}_{1-\tau}(x) = \left[\widehat{\rho}(x) \pm \frac{s_n(x)}{x^2 \sqrt{n} h \Delta} c_n^G(1 - \tau) \right], \quad x \in I.$$

Since $\rho(x) \in \widehat{\mathcal{C}}_{1-\tau}(x) \forall x \in I \Leftrightarrow \|\sqrt{n}\Delta h x^2(\hat{\rho} - \rho)/s_n\|_I \leq c_n^G(1 - \tau)$, it is seen that

$$\mathbb{P}\{\rho(x) \in \widehat{\mathcal{C}}_{1-\tau}(x) \forall x \in I\} = \mathbb{P}\{\|Z_n^G\|_I \leq c_n^G(1 - \tau)\} + o(1) \geq 1 - \tau + o(1),$$

so that $\widehat{\mathcal{C}}_{1-\tau}$ will be a valid confidence band for ρ on I with level approximately $1 - \tau$, provided that (3.3) holds.

Now, we shall estimate the quantile $c_n^G(1 - \tau)$, in addition to the variance function $s_n^2(x)$. The latter can be estimated from

$$\widehat{s}_n^2(x) = \frac{1}{n} \sum_{j=1}^n Y_{n,j}^4 \widehat{K}_n^2((x - Y_{n,j})/h) - \left\{ \frac{1}{n} \sum_{j=1}^n Y_{n,j}^2 \widehat{K}_n((x - Y_{n,j})/h) \right\}^2, \quad x \in I,$$

where \widehat{K}_n is the function defined by

$$\widehat{K}_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{\varphi_W(u)}{\widehat{\varphi}_\Delta(u/h)} du, \quad x \in \mathbb{R}.$$

Note that as long as $h \gtrsim \Delta^{1/2}$, $\inf_{|u| \leq h^{-1}} |\widehat{\varphi}_\Delta(u)| \gtrsim 1 - o_{\mathbb{P}}(1)$, so that \widehat{K}_n is well-defined with probability approaching one. To estimate the quantile $c_n^G(1 - \tau)$, we consider two bootstrap methods. The one is the Gaussian multiplier (or wild) bootstrap, and the other is the empirical (or Efron's) bootstrap.

Gaussian multiplier bootstrap. Generate $\xi_1, \dots, \xi_n \sim N(0, 1)$ i.i.d., independent of the data $\mathcal{D}_n = \{Y_{n,j}\}_{j=1}^n$, and construct the multiplier process

$$\widehat{Z}_n^{MB}(x) = \frac{1}{\widehat{s}_n(x)\sqrt{n}} \sum_{j=1}^n \xi_j \left\{ Y_{n,j}^2 \widehat{K}_n((x - Y_{n,j})/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 \widehat{K}_n((x - Y_{n,j'})/h) \right\}, \quad x \in I,$$

where $\widehat{s}_n(x) = \sqrt{\widehat{s}_n^2(x)}$. Note that under the regularity conditions stated below, $\inf_{x \in I} \widehat{s}_n^2(x) \gtrsim (1 - o_p(1))\Delta h$, so that \widehat{Z}_n^{MB} is well-defined with probability approaching one. Conditionally on the data \mathcal{D}_n , \widehat{Z}_n^{MB} is a Gaussian process whose covariance function “estimates” that of Z_n^G . Hence, we estimate $c_n^G(1 - \tau)$ by

$$\widehat{c}_n^{MB}(1 - \tau) = \text{conditional } (1 - \tau)\text{-quantile of } \|\widehat{Z}_n^{MB}\|_I \text{ given } \mathcal{D}_n,$$

which can be computed via simulations. The resulting confidence band is

$$\widehat{\mathcal{C}}_{1-\tau}^{MB}(x) = \left[\widehat{\rho}(x) \pm \frac{\widehat{s}_n(x)}{x^2 \sqrt{n} h \Delta} \widehat{c}_n^{MB}(1 - \tau) \right], \quad x \in I.$$

Empirical bootstrap. Next, we consider the empirical bootstrap. Let $P_{n,\Delta} = n^{-1} \sum_{j=1}^n \delta_{Y_{n,j}}$ denote the empirical distribution. Conditionally on the data, generate $Y_{n,1}^b, \dots, Y_{n,n}^b \sim P_{n,\Delta}$ i.i.d., and construct the bootstrap process

$$\widehat{Z}_n^{EB}(x) = \frac{1}{\widehat{s}_n(x)\sqrt{n}} \sum_{j=1}^n \left\{ (Y_{n,j}^b)^2 \widehat{K}_n((x - Y_{n,j}^b)/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 \widehat{K}_n((x - Y_{n,j'})/h) \right\}, \quad x \in I.$$

We estimate $c_n^G(1 - \tau)$ by

$$\widehat{c}_n^{EB}(1 - \tau) = \text{conditional } (1 - \tau)\text{-quantile of } \|\widehat{Z}_n^{EB}\|_I \text{ given } \mathcal{D}_n.$$

The resulting confidence band is

$$\widehat{\mathcal{C}}_{1-\tau}^{EB}(x) = \left[\widehat{\rho}(x) \pm \frac{\widehat{s}_n(x)}{x^2 \sqrt{n} h \Delta} \widehat{c}_n^{EB}(1 - \tau) \right], \quad x \in I.$$

Remark 3.1 (Scaling by $1/x^2$). One might think that, because of the scaling by $1/x^2$, our confidence bands would be too wide if x is close to the origin. However, heuristically, the standard deviation function $s_n(x)$ would scale like $x^2 \sqrt{\Delta h \rho(x)}$ for $x \neq 0$, so the scaling factor $1/x^2$ would be canceled out and $s_n(x)/(x^2 \sqrt{n} \Delta h)$ would scale like $\sqrt{\rho(x)/(n \Delta h)}$. To see this, assuming that ρ has finite fourth moment (which ensures that P_Δ has finite fourth moment), observe that since $y^4 P_\Delta(dy)/\Delta \rightarrow y^4 \rho(y) dy$ weakly as finite measures³, we have that, heuristically,

$$\begin{aligned} s_n^2(x) &\approx \mathbb{E}[Y_{n,1}^4 K_n^2((x - Y_{n,1})/h)] \\ &\approx \Delta \int_{\mathbb{R}} y^4 K_n^2((x - y)/h) \rho(y) dy \\ &\approx \Delta h x^4 \rho(x) \int_{\mathbb{R}} K_n^2(y) dy. \end{aligned}$$

³To see this, observe that, for each $u \in \mathbb{R}$, $\varphi_\Delta^{(4)}(u) = \Delta \psi^{(4)}(u) \varphi_\Delta(u) + o(\Delta) = \Delta \int_{\mathbb{R}} e^{iuy} y^4 \rho(y) dy + o(\Delta)$, so that

$$\int_{\mathbb{R}} e^{iuy} \frac{y^4 P_\Delta(dy)}{\Delta} = \frac{\varphi_\Delta^{(4)}(u)}{\Delta} \rightarrow \int_{\mathbb{R}} e^{iuy} y^4 \rho(y) dy,$$

which implies that $y^4 P_\Delta(dy)/\Delta \rightarrow y^4 \rho(y) dy$ weakly.

Of course, these approximations are only heuristic, but the discussion so far at least provides a partial explanation for that the scaling factor $1/x^2$ would not too much inflate the width of our bands. See also figures in Section 6.

4. MAIN RESULTS

4.1. Validity of bootstrap confidence bands. In this section, we prove validity of the proposed confidence bands $\widehat{\mathcal{C}}_{1-\tau}^{MB}$ and $\widehat{\mathcal{C}}_{1-\tau}^{EB}$. To this end, we make the following assumption. Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be α -Hölder continuous for $\alpha \in (0, 1]$ if

$$\sup_{x, y \in \mathbb{R}, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

Let P_Δ denote the distribution of $L_\Delta = Y_{n,1}$, and for any $b \in \mathbb{R}$, let $P_{\Delta,b}$ denote the distribution of $L_\Delta - b\Delta$. Let I be a compact set in $\mathbb{R} \setminus \{0\}$.

Assumption 4.1. *We assume the following conditions.*

- (i) $\int_{\mathbb{R}} |x|^{4+\zeta} \rho(x) dx < \infty$ for some $\zeta > 0$.
- (ii) For some $b \in \mathbb{R}$, the measure $y^4 P_{\Delta,b}(dy)$ has Lebesgue density $g_{\Delta,b}$ such that $\|g_{\Delta,b}\|_{\mathbb{R}} \lesssim \Delta$ and $\inf_{y \in I^{\varepsilon_0}} g_{\Delta,b}(y) \gtrsim \Delta$ for some sufficiently small $\varepsilon_0 > 0$ with $0 \notin I^{\varepsilon_0}$.
- (iii) Let $r > 0$, and let p be the integer such that $p < r \leq p + 1$. The function $x^2 \rho$ is p -times differentiable, and $(x^2 \rho)^{(p)}$ is $(r - p)$ -Hölder continuous.
- (iv) Let $W : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function (kernel) such that

$$\left\{ \begin{array}{l} \int_{\mathbb{R}} W(x) dx = 1, \quad \int_{\mathbb{R}} |x|^{p+1} |W(x)| dx < \infty, \\ \int_{\mathbb{R}} x^\ell W(x) dx = 0, \quad \ell = 1, \dots, p, \\ \varphi_W \text{ is } C^4 \text{ and } \varphi_W(u) = 0 \quad \forall |u| > 1, \end{array} \right. \quad (4.1)$$

where φ_W is the Fourier transform of W .

- (v) $h \gtrsim \Delta^{1/2}$, $n^{1/2-\delta} \sqrt{\Delta h} \rightarrow \infty$ for some $\delta \in (0, \frac{r}{2r+3})$, and $h^r \sqrt{n \Delta h \log n} \rightarrow 0$.
- (vi) Let $\widehat{\sigma}^2$ be an estimator of σ^2 such that $|\widehat{\sigma}^2 - \sigma^2| \cdot \|h^{-1} W(\cdot/h)\|_I = o_{\mathbb{P}}\{(n \Delta h \log n)^{-1/2}\}$.

Condition (i) is a moment condition and is equivalent to finiteness of the $(4 + \zeta)$ -th moment of L_1 (and L_t for all $t > 0$; see [58], Corollary 25.8). Condition (i) excludes, e.g., α -stable processes for $\alpha \in (0, 2)$, but it allows for ρ not to be integrable (i.e., $\nu(\mathbb{R}) = \infty$ is allowed). Condition (ii) is a high-level condition and will be discussed in detail in the next subsection. However, at this point, we would like to remark that Condition (ii) is satisfied by a wide class of Lévy processes. Importantly, Condition (ii) allows the distribution P_Δ to have a discrete component. For example, if $L_t = bt + J_t$ where $J = (J_t)_{t \geq 0}$ is a compound Poisson process (with absolutely continuous jump size distribution), then P_Δ has a point mass at $b\Delta$ and $P_{\Delta,b} = P_{\Delta,b}(\{0\})\delta_0 + P_{\Delta,b}^{ac}$ where $P_{\Delta,b}^{ac}$ is absolutely continuous. In this case, P_Δ itself is not absolutely continuous, but $y^4 P_{\Delta,b} = y^4 P_{\Delta,b}^{ac}$ is absolutely continuous.

Condition (iii) is concerned with smoothness of the scaled Lévy density $x^2 \rho$. Condition (iii) allows the Lévy density to have a ‘‘cusp’’ at the origin. For example, a Gamma process has Lévy density $\rho(x) = \alpha x^{-1} e^{-\beta x} \mathbf{1}_{(0, \infty)}(x)$ for some $\alpha, \beta > 0$; in this case, the Lévy density itself ρ is discontinuous (at the origin), but the scaled version $x^2 \rho$ is globally Lipschitz continuous. Condition (iv) is concerned with the kernel function W . We assume that W is a $(p + 1)$ -th order kernel, but allow for the possibility that $\int_{\mathbb{R}} x^{p+1} W(x) dx = 0$. We will provide examples

of kernel functions satisfying Condition (iv) in Remark 4.2 below. It is worth mentioning that since the Fourier transform of W has compact support, the support of the kernel function W itself is necessarily unbounded (which is a consequence of the Paley-Wiener theorem; see [61], Theorem 4.1), and we will use global regularity of the scaled Lévy density $x^2\rho$ to suitably bound the deterministic bias, despite that we focus on constructing confidence bands on a compact set that does not intersect the origin. It could be possible to replace Condition (iii) by a “local” smoothness condition on $x^2\rho$, but we shall keep current Condition (iii) for the simplicity of the exposition.

Condition (v) is concerned with the bandwidth and the time span Δ . The condition $h \gtrsim \Delta^{1/2}$ ensures that $\inf_{|u| \leq h^{-1}} |\varphi_\Delta(u)| \gtrsim 1$ (see Lemma 7.2). The condition $h^r \sqrt{n\Delta h \log n} \rightarrow 0$ is an “undersmoothing” condition. Inspection of the proof of Theorem 4.1 shows that, without the condition $h^r \sqrt{n\Delta h \log n} \rightarrow 0$, we have that

$$\|\widehat{\rho} - \rho\|_I = O_{\mathbb{P}}\{(n\Delta h)^{-1/2} \sqrt{\log n}\} + O(h^r),$$

where the $O(h^r)$ term comes from the deterministic bias. The right hand side is optimized by choosing $h \sim (n\Delta/\log n)^{-1/(2r+1)}$, and the optimal rate for $\|\widehat{\rho} - \rho\|_I$ is $(n\Delta/\log n)^{-r/(2r+1)}$. For our confidence bands to be valid, however, we have to choose bandwidths of smaller order (by $\log n$ factors) than the optimal one for estimation under the sup-norm, so that the bias term is negligible relative to the “variance” or stochastic term. Undersmoothing bandwidths are commonly used in construction of confidence bands. See Section 5.7 in [65] for related discussions. For example, if we choose $h \sim (n\Delta)^{-1/(2r+1)}(\log n)^{-1}$, then Condition (v) reduces to

$$n^{(2r+1)\delta/r-1}(\log n)^{(2r+1)/(2r)} \ll \Delta \lesssim n^{-1/(r+3/2)}(\log n)^{-(2r+1)/(r+3/2)}, \quad (4.2)$$

where the condition $\delta \in (0, \frac{r}{2r+3})$ ensures that $(2r+1)\delta/r - 1 < -1/(r+3/2)$.

Condition (vi) is concerned with the pilot estimator of σ^2 . Since the set I is away from the origin, if $|W(x)| = O(|x|^{-r-1})$ as $|x| \rightarrow \infty$, then $\|h^{-1}W(\cdot/h)\|_I = O(h^r) = o\{(n\Delta h \log n)^{-1/2}\}$, so that the pilot estimator $\widehat{\sigma}^2$ need not be even consistent (e.g. we may take $\widehat{\sigma}^2 = 0$). Note that as long as $\int_{\mathbb{R}} |x|^{p+2}|W(x)|dx < \infty$, the Fourier transform φ_W is $(p+2)$ -times continuously differentiable, so that $|W(x)| = o(|x|^{-p-2}) = o(|x|^{-r-1})$ as $|x| \rightarrow \infty$ (however, as noted before, in our simulations studies, we found that, when $\sigma > 0$, using a proper estimator for $\widehat{\sigma}^2$ improves upon the empirical performance of the estimator $\widehat{\rho}$ and the confidence bands).

The following theorem derives a Gaussian approximation result. Recall that a Gaussian process $\{Z(x) : x \in I\}$ is a tight random variable in $\ell^\infty(I)$ if and only if I is totally bounded for the intrinsic pseudo-metric $d_Z(x, y) = \sqrt{\mathbb{E}[(Z(x) - Z(y))^2]}$ for $x, y \in I$, and Z has sample paths almost surely uniformly d_Z -continuous [cf. 63, p.41]. In that case, we say that Z is a tight Gaussian random variable in $\ell^\infty(I)$.

Theorem 4.1 (Gaussian approximation). *Under Assumption 4.1, for each sufficiently large n , there exists a tight Gaussian random variable Z_n^G in $\ell^\infty(I)$ with mean zero and covariance function*

$$\begin{aligned} \text{Cov}(Z_n^G(x), Z_n^G(y)) &= \frac{1}{s_n(x)s_n(y)} \left\{ \int_{\mathbb{R}} K_n((x-w)/h)K_n((y-w)/h)w^4 P_\Delta(dw) \right. \\ &\quad \left. - \left(\int_{\mathbb{R}} K_n((x-w)/h)w^2 P_\Delta(dw) \right) \left(\int_{\mathbb{R}} K_n((y-w)/h)w^2 P_\Delta(dw) \right) \right\} \end{aligned}$$

for $x, y \in I$, and such that as $n \rightarrow \infty$,

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|\sqrt{n}\Delta h x^2(\hat{\rho} - \rho)/s_n\|_I \leq z\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \rightarrow 0.$$

The distribution of the Gaussian process Z_n^G that appears in Theorem 4.1 changes with n , and so the approximation is only an “intermediate” one. However, such an intermediate Gaussian approximation is sufficient to prove validity of bootstraps [cf. 16]. Building on Theorem 4.1, the following theorem formally establishes asymptotic validity of the two bootstrap confidence bands.

Theorem 4.2 (Validity of bootstrap confidence bands). *Under Assumption 4.1, for either $B \in \{MB, EB\}$, we have*

$$\mathbb{P}\{\rho(x) \in \widehat{\mathcal{C}}_{1-\tau}^B(x) \ \forall x \in I\} \rightarrow 1 - \tau$$

as $n \rightarrow \infty$. Furthermore, the supremum width of the band $\widehat{\mathcal{C}}_{1-\tau}^B$ is $O_{\mathbb{P}}\{(n\Delta h)^{-1/2}\sqrt{\log n}\}$.

Remark 4.1. For example, if we choose $h \sim (n\Delta)^{-1/(2r+1)}(\log n)^{-1}$, then provided that (4.2) is satisfied, the supremum width of the band $\widehat{\mathcal{C}}_{1-\tau}^B$ is $O_{\mathbb{P}}\{(n\Delta)^{-r/(2r+1)} \log n\}$.

The proofs of Theorems 4.1 and 4.2 build on non-trivial applications of the intermediate Gaussian and bootstrap approximation theorems developed in [15, 16, 18]. We would like to point out here that there are several non-trivial steps in proving Theorems 4.1 and 4.2. For example, we will require to show that $\inf_{x \in I} s_n^2(x) \gtrsim \Delta h$, but since the increment distribution P_Δ may have a discrete component and degenerates to δ_0 as $\Delta \rightarrow 0$, and K_n changes with n and has unbounded support, lower bounding the variance function $s_n^2(x)$ is non-trivial. Second, a crucial fact in the proofs of Theorems 4.1 and 4.2 is that the function class

$$\{y \mapsto y^2 K_n((x - y)/h) : x \in I\} \quad (4.3)$$

is a Vapnik-Chervonenkis (VC) type class. In view of Lemma 1 in [45], it is not difficult to verify that the function class (4.3) is VC type for an envelope function of the form $y \mapsto \text{const.} \times y^2$; however, using this envelope function will require more restrictive moment conditions on ρ (we will require at least finite eighth moment of ρ) and additional conditions on the smoothness level r depending on the moment conditions on ρ . In fact, although it is not apparent, it turns out that, under our assumption, the function $y \mapsto y^2 K_n((x - y)/h)$ is bounded uniformly in n and $x \in I$. So, we will verify that the function class (4.3) is VC type for a constant envelope function, which requires a different and non-trivial idea; cf. Lemma 7.7.

Remark 4.2 (Examples of kernel functions). Construction of a kernel function satisfying Condition (iv) is typically done by specifying its Fourier transform φ_W . Let $\varpi : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is even (i.e., $\varpi(u) = \varpi(-u)$), supported in $[-1, 1]$, and $(4 \vee (p + 3))$ -times continuously differentiable, and such that

$$\varpi^{(\ell)}(0) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell = 1, \dots, p \end{cases}.$$

Then the function $W(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \varpi(u) du$ is real-valued, $|W(x)| = o(|x|^{-p-3})$ as $|x| \rightarrow \infty$ (which follows from changes of variables), so that $(1 \vee |x|^{p+1})W$ is integrable, and

$$\int_{\mathbb{R}} x^\ell W(x) dx = i^{-\ell} \varpi^{(\ell)}(0) = \begin{cases} 1 & \ell = 0 \\ 0 & \ell = 1, \dots, p \end{cases}.$$

Here, since W is even, if p is even, we also have $\int_{\mathbb{R}} x^{p+1}W(x)dx = 0$. Examples of ϖ include: $\varpi(u) = (1 - u^2)^k 1_{[-1,1]}(u)$ for $k \geq 5 \vee (p + 4)$, and

$$\varpi(u) = \begin{cases} 1 & \text{if } |u| \leq c \\ \exp\left\{\frac{-b \exp(-b/(|u|-c)^2)}{(|u|-1)^2}\right\} & \text{if } c < |u| < 1 \\ 0 & \text{if } 1 \leq |u| \end{cases} \quad (4.4)$$

for $0 < c < 1$ and $b > 0$. For the latter case, ϖ is infinitely differentiable with $\varpi^{(\ell)}(0) = 0$ for all $\ell \geq 1$, so that its inverse Fourier transform W , called a flap-top kernel, is of infinite order, i.e., $\int_{\mathbb{R}} x^\ell W(x)dx = 0$ for all integers $\ell \geq 1$ [cf. 51].

4.2. Discussions on Condition (ii) in Assumption 4.1. In this subsection, we provide primitive regularity conditions that guarantee Condition (ii) in Assumption 4.1. We make the following assumption.

Assumption 4.2. *Assume that $x^2\rho$ is continuous; $x^4\rho \in \ell^\infty(\mathbb{R})$; and ρ is positive on I^{ε_1} for some sufficiently small $\varepsilon_1 > 0$ such that $0 \notin I^{\varepsilon_1}$. Furthermore, there exists $b \in \mathbb{R}$ such that the signed measure $yP_{\Delta,b}(dy)$ has a Lebesgue density bounded (in absolute value) by $C \log(1/\Delta)$ for all sufficiently small $\Delta > 0$ for some constant $C > 0$.*

Assumption 4.2 ensures Condition (ii) in Assumption 4.1 to hold.

Proposition 4.1. *Condition (ii) in Assumption 4.1 is satisfied under Assumption 4.2.*

Remark 4.3. The first part of Assumption 4.2 is not restrictive. Recall that we are assuming finiteness of the $(4 + \zeta)$ -th moment of ρ in Assumption 4.1, so that the requirement that $x^4\rho \in \ell^\infty(\mathbb{R})$ appears to be innocuous. Proposition 16 in [56] provides primitive regularity conditions that ensure that yP_Δ has a density bounded uniformly in Δ ; see Assumption 15 in [56] (we allow for the density of $yP_{\Delta,b}(dy)$ to grow like $\log(1/\Delta)$ to cover cases where $\sigma = 0$ and the Lévy density behaves like x^{-2} near the origin; see below). In particular, Assumption 15 in [56] covers many of basic examples of Lévy processes.

For example, consider the following two simple cases:

- (a) $\sigma > 0$; or (b) $\sigma = 0$ and $\|x\rho\|_{\mathbb{R}} < \infty$.

($\|x\rho\|_{\mathbb{R}} < \infty$ together with the assumption that $\int_{\mathbb{R}} x^2\rho(x)dx < \infty$ ensure that $\int_{\mathbb{R}} |x|\rho(x) < \infty$.)

For Case (a), in view of $\varphi'_\Delta(u) = i \int_{\mathbb{R}} e^{iuy} y P_\Delta(dy)$ together with the fact that φ'_Δ is integrable, a Lebesgue density of yP_Δ exists and is given by

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuy} i^{-1} \varphi'_\Delta(u) du.$$

Since $|\varphi'_\Delta(u)| \leq \text{const.} \Delta(1 + |u|)e^{-\Delta\sigma^2 u^2/2}$, we see that

$$\int_{\mathbb{R}} |\varphi'_\Delta(u)| du \lesssim \Delta \int_{\mathbb{R}} (1 + |u|) e^{-\Delta\sigma^2 u^2/2} du \lesssim 1,$$

which shows that the density of yP_Δ is bounded uniformly in Δ .

For Case (b), observe that $\psi(u) = iub + \int_{\mathbb{R}} (e^{iux} - 1)\rho(x)dx$ with $b = \gamma_c - \int_{\mathbb{R}} x\rho(x)dx$, and

$$\int_{\mathbb{R}} e^{iuy} y P_{\Delta,b}(dy) = i^{-1} (\mathbb{E}[e^{iu(L_\Delta - b\Delta)}])' = \Delta \left(\int_{\mathbb{R}} e^{iuy} y \rho(y) dy \right) \mathbb{E}[e^{iu(L_\Delta - b\Delta)}].$$

Applying the Fourier inversion, we see that $yP_{\Delta,b}$ has a Lebesgue density

$$\Delta \int_{\mathbb{R}} (y-w)\rho(y-w)P_{\Delta,b}(dw),$$

which is bounded (in absolute value) by $\Delta\|x\rho\|_{\mathbb{R}}P_{\Delta,b}(\mathbb{R}) \lesssim \Delta$. Hence, in these two cases, $yP_{\Delta,b}$ has a Lebesgue density bounded uniformly in Δ for some $b \in \mathbb{R}$. For other more complicated cases, we refer to Proposition 16 in [56].

For the symmetric tempered stable process with stability index $\alpha = 1$ and the Normal Inverse Gaussian process discussed in the next section, whose Lévy densities behave like x^{-2} near the origin, Proposition 16 in [56] appears not to be directly applicable. To cover those cases, we present the following lemma.

Lemma 4.1. *Suppose that $\sigma = 0$ and the Lévy density ρ satisfies that for some constants $C > c > 0$,*

$$c \leq \frac{1}{\varepsilon} \int_{|x| \leq \varepsilon} x^2 \rho(x) dx \leq C \quad \text{and} \quad \int_{|x| > \varepsilon} |x| \rho(x) dx \leq C(1 + \log(1/\varepsilon))$$

for all $\varepsilon \in (0, 1)$. Then the signed measure $yP_{\Delta}(dy)$ has a Lebesgue density bounded (in absolute value) by $C' \log(1/\Delta)$ for all sufficiently small $\Delta > 0$ for some constant $C' > 0$.

Next, we discuss lower bounding increment densities, which is related to verification of the second part of Condition (ii).

Remark 4.4 (Lower bounding increment densities). Suppose that the increment distribution P_{Δ} is absolutely continuous with density p_{Δ} (which, in view of Theorem 27.7 in [58], only excludes the compound Poisson case). To verify the second part of Condition (ii), it suffices to verify that the infimum of p_{Δ} on a small neighborhood of I is $\gtrsim \Delta$ (we have assumed here that $b = 0$, which does not lose generality since $P_{\Delta,b}$ corresponds to the increment distribution of another Lévy process $L_t - bt$ that shares the same Lévy density as L). Now, it is known that, under a number of technical conditions, the increment density p_{Δ} can be expanded as

$$p_{\Delta}(y) = \Delta\rho(y) + o(\Delta), \tag{4.5}$$

where $o(\Delta)$ is uniform in y outside a neighborhood of the origin. See [57] and [32]. Provided that the expansion (4.5) is valid and $\rho(y)$ is bounded away from zero on a neighborhood of I , it is immediate to see that p_{Δ}/Δ is bounded away from zero on a small neighborhood of I . However, the expansion (4.5) is known to be valid only under restrictive conditions including sufficient smoothness of the Lévy density (outside the origin) and certain high-level conditions; see the aforementioned references. So, in verification of the second part of Condition (ii), we do not rely on such expansion results and provide a direct proof. The proof of Proposition 4.1 shows that the second part of Condition (ii) is in fact valid under quite mild conditions.

5. EXAMPLES

In this section, we provide some examples of Lévy processes that satisfy Conditions (i)-(iii) in Assumption 4.1. For detailed properties of Lévy processes discussed below, we refer to [25]. The first four examples are purely non-Gaussian Lévy processes (i.e., $\sigma = 0$), and we allow them to have drift terms.

Example 5.1 (Compound Poisson process). A compound Poisson process with drift is a stochastic process of the form $L_t = bt + \sum_{k=1}^{N_t} X_k$, where $N = (N_t)_{t \geq 0}$ is a Poisson process with constant intensity $\lambda > 0$ and $\{X_k\}_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with common distribution F_X (jump size distribution) independent of N . We assume that F_X has Lebesgue density f_X . In this case, the characteristic exponent is $\psi(u) = iub + \int_{\mathbb{R}} (e^{iux} - 1) \lambda f_X(x) dx$, and so the Lévy density is $\rho = \lambda f_X$. Conditions (i) and (iii) can be directly translated to conditions on f_X . From Proposition 4.1 and Remark 4.3, Condition (ii) is satisfied if $x^2 f_X$ is continuous, $(|x| \vee x^4) f_X \in \ell^\infty(\mathbb{R})$, and f_X is positive on I^{ε_1} for some $\varepsilon_1 > 0$ such that $0 \notin I^{\varepsilon_1}$.

For example, the jump-part of the Merton model [52] is a compound Poisson process with jump size distribution $N(c, v^2)$ for some $c \in \mathbb{R}$ and $v^2 > 0$, for which it is not difficult to verify that Conditions (i)-(iii) are satisfied with arbitrarily large $r > 0$, and any compact set I in $\mathbb{R} \setminus \{0\}$. The jump-part of the Kou model [49] is a compound Poisson process with jump size density

$$f_X(x) = p\lambda_+ e^{-\lambda_+ x} 1_{(0, \infty)}(x) + (1-p)\lambda_- e^{-\lambda_- |x|} 1_{(-\infty, 0)}(x)$$

for some $p \in [0, 1]$ and $\lambda_+, \lambda_- > 0$. Let I be any compact set in $\mathbb{R} \setminus \{0\}$, $(0, \infty)$, and $(-\infty, 0)$ if $0 < p < 1$, $p = 1$, and $p = 0$, respectively. Then, it is not difficult to verify that Conditions (i)-(iii) are satisfied with $r = 3$ if $p = 1/2$ and $\lambda_+ = \lambda_-$, and $r = 2$ otherwise.

A compound Poisson process is a process of finite activity, i.e., has only finitely many jumps on any bounded time interval.

Example 5.2 (Tempered stable process). A tempered stable process with index $0 \leq \alpha < 2$ is a Lévy process with Lévy density

$$\rho(x) = \frac{c_+}{x^{1+\alpha}} e^{-\lambda_+ x} 1_{(0, \infty)}(x) + \frac{c_-}{|x|^{1+\alpha}} e^{-\lambda_- |x|} 1_{(-\infty, 0)}(x),$$

where $c_+, c_- \geq 0$, $c_+ + c_- > 0$, and $\lambda_+, \lambda_- > 0$. We assume that the stability index is restricted to $0 \leq \alpha < 1$. Let I be any compact set in $\mathbb{R} \setminus \{0\}$, $(0, \infty)$, and $(-\infty, 0)$ if $c_+ c_- \neq 0$, $c_- = 0$, and $c_+ = 0$, respectively. It is clear that Conditions (i) and (iii) are satisfied with $r = 2$ if $\alpha = 0$, $c_+ = c_-$, and $\lambda_+ = \lambda_-$, and $r = 1 - \alpha$ otherwise. For example, to see that $x^2 \rho$ is $(1 - \alpha)$ -Hölder continuous in the latter case, it suffices to verify that $x^{1-\alpha} e^{-x}$ is $(1 - \alpha)$ -Hölder continuous on $[0, \infty)$, which can be verified as follows: for any $y > x \geq 0$,

$$|x^{1-\alpha} e^{-x} - y^{1-\alpha} e^{-y}| \leq \underbrace{x^{1-\alpha} e^{-x}}_{\text{bounded}} (1 - e^{-(y-x)}) + \underbrace{e^{-y}}_{\leq 1} \underbrace{(y^{1-\alpha} - x^{1-\alpha})}_{\leq (y-x)^{1-\alpha}} \leq \text{const.} \times (y-x)^{1-\alpha},$$

where we have used that $1 - e^{-(y-x)} \leq (1 - e^{-(y-x)})^{1-\alpha} \leq (y-x)^{1-\alpha}$.

If $\alpha = 0$, then since $x\rho$ is bounded, in view of Remark 4.3, $yP_{\Delta, b}$ with $b = \gamma_c - \int_{\mathbb{R}} x\rho(x) dx$ has a Lebesgue density bounded uniformly in Δ . If $0 < \alpha < 1$, then Assumption 15 Case (iii) in [56] is satisfied for $L_t - bt$ with $b = \gamma_c - \int_{\mathbb{R}} x\rho(x) dx$, and hence Proposition 16 in [56] yields that $yP_{\Delta, b}$ has a Lebesgue density bounded uniformly in Δ . Therefore, by Proposition 4.1, Condition (ii) is satisfied in either case.

The tempered stable process includes Gamma, Inverse Gaussian, and Variance Gamma processes as special cases. A Gamma process corresponds to the case with $\alpha = 0$ and $c_- = 0$; an Inverse Gaussian process corresponds to the case with $\alpha = 1/2$ and $c_- = 0$; and a Variance Gamma process corresponds to the case with $\alpha = 0$ and $c_+ = c_- > 0$.

The tempered stable process is a process of infinite activity, i.e., has infinitely many jumps on any bounded time interval.

The Lévy density in each of Examples of 5.1 and 5.2 has finite first moment, and therefore sample paths of the process have finite variation on any bounded time interval [58, Theorem 21.9]. On the other hand, the following two examples have infinite variation on any bounded time interval.

Example 5.3 (Symmetric tempered stable process with $\alpha = 1$). In the previous example, consider the case where $\alpha = 1, c_+ = c_- = c > 0$, and $\lambda_+ = \lambda_- = \lambda > 0$, i.e.,

$$\rho(x) = \frac{c}{x^2} e^{-\lambda|x|}, \quad x \neq 0.$$

In this case, Condition (i) is trivially satisfied, and since $x^2\rho(x)$ extends to a Lipschitz continuous function on \mathbb{R} as $x^2\rho(x) = ce^{-\lambda|x|}, x \in \mathbb{R}$, Condition (iii) is satisfied with $r = 1$. It is not difficult to verify that the assumption in Lemma 4.1 is satisfied, and therefore, by Proposition 4.1, Condition (ii) is satisfied for any compact set I in $\mathbb{R} \setminus \{0\}$.

Example 5.4 (Normal Inverse Gaussian process). A Normal Inverse Gaussian (NIG) process is a purely non-Gaussian Lévy process with Lévy density

$$\rho(x) = \frac{\delta\alpha e^{\beta x}}{\pi|x|} \mathbb{K}_1(\alpha|x|), \quad x \neq 0$$

where \mathbb{K}_1 is the modified Bessel function of the second kind with order 1, and has integral representation

$$\mathbb{K}_1(z) = \frac{1}{2} \int_0^\infty e^{-\frac{z}{2}(t+t^{-1})} dt, \quad z > 0.$$

The parameters α, β, δ are restricted such that $0 \leq |\beta| < \alpha$ and $\delta > 0$. Since $\mathbb{K}_1(z)$ decays like $z^{-1/2}e^{-z}$ as $z \rightarrow +\infty$, Condition (i) is satisfied. By a change of variables, we have that

$$\rho(x) = \frac{\delta e^{\beta x}}{\pi x^2} \int_0^\infty e^{-t - \frac{\alpha^2 x^2}{4t}} dt, \quad x \neq 0.$$

Since the integral on the right hand side is well-defined for $x = 0$, $x^2\rho(x)$ extends to a continuous function on \mathbb{R} as

$$\rho_{\sharp}(x) := x^2\rho(x) = \frac{\delta e^{\beta x}}{\pi} \int_0^\infty e^{-t - \frac{\alpha^2 x^2}{4t}} dt, \quad x \in \mathbb{R}. \quad (5.1)$$

Furthermore, it is not difficult to verify that the assumption in Lemma 4.1 is satisfied, and therefore, by Proposition 4.1, Condition (ii) is satisfied for any compact set I in $\mathbb{R} \setminus \{0\}$.

Finally, it appears to be difficult to directly verify Condition (iii) to the NIG process, but inspection of the proofs of Theorems 4.1 and 4.2 shows that Condition (iii) is used only to bound the deterministic bias $\|\rho_{\sharp} * (h^{-1}W(\cdot/h)) - \rho_{\sharp}\|_I$. Fortunately, for the NIG process, it is possible to directly bound the deterministic bias $\|\rho_{\sharp} * (h^{-1}W(\cdot/h)) - \rho_{\sharp}\|_I \lesssim h^r$ for any $r \in (0, 1)$; see below. Therefore, the conclusions of Theorems 4.1 and 4.2 hold true for the NIG process with any $r \in (0, 1)$, provided that other technical conditions (Conditions (iv)-(vi)) are satisfied.

Lemma 5.1. *Let ρ_{\sharp} be as in (5.1), and let $W : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function such that $\int_{\mathbb{R}} W(x)dx = 1$ and $\int_{\mathbb{R}} x^2|W(x)|dx < \infty$. Then for any $r \in (0, 1)$ and any nonempty compact set I in \mathbb{R} , we have that $\|\rho_{\sharp} * (h^{-1}W(\cdot/h)) - \rho_{\sharp}\|_I \lesssim h^r$, where $*$ denotes the convolution.*

Example 5.5 (Brownian motion + purely non-Gaussian Lévy process). Let $L_t = \sigma B_t + J_t$, where $\sigma > 0$, $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, and $J = (J_t)_{t \geq 0}$ is a purely non-Gaussian Lévy process independent of B (with drift). We assume that J is one of purely non-Gaussian Lévy processes in Examples 5.1–5.4. For the compound Poisson process case, we assume that $x^4 f_X \in \ell^\infty(\mathbb{R})$, and Conditions (i) and (iii) are satisfied for $\rho = \lambda f_X$. In view of Remark 4.3, it is clear that Conditions (i)–(iii) are satisfied with r given in Examples 5.1–5.4, as long as I is properly chosen.

6. SIMULATION RESULTS

6.1. Simulation framework. In this section, we present simulation results to evaluate the finite-sample performance of the proposed confidence bands. We consider the following data generating processes.

- (1) **Brownian motion + compound Poisson process.** Let $L_t = \sigma B_t + J_t$, where $\sigma \geq 0$, $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, and $J_t = \sum_{k=1}^{N_t} X_k$ is a compound Poisson process (see Example 5.1). The Poisson process N has intensity $\lambda > 0$. We set $\lambda = 4$ or 10. We consider two types of jump size distributions. For the first case, f_X is the density of the normal distribution $N(0, v^2)$ and the Lévy density is $\rho(x) = \lambda e^{-x^2/(2v^2)} / \sqrt{2\pi v^2}$. We denote this case by $\text{BCN}(\sigma, v)$. For the second case, f_X is the density of the Laplace distribution with location 0 and scale $v > 0$, and the Lévy density is $\rho(x) = \lambda e^{-|x|/v} / (2v)$. We denote the latter case by $\text{BCL}(\sigma, v)$. For $\text{BCN}(\sigma, v)$ and $\text{BCL}(\sigma, v)$, we take $I = [-0.75, -0.25] \cup [0.25, 0.75]$.

If $\sigma = 0$, then L reduces to the compound Poisson process J , for which we set $\hat{\sigma}^2 = 0$. In case of $\sigma > 0$, we estimate σ^2 by the TRV estimator $\hat{\sigma}_{TRV}^2$ with $\alpha_0 = 3$ and $\theta_0 = 0.48$ (see Example 2.1). We also examined the performance of the confidence bands with estimated σ^2 in case of $\sigma = 0$, but the simulation results are almost identical to those under $\hat{\sigma}^2 = 0$. Hence, we only report the simulation results with $\hat{\sigma}^2 = 0$ in case of $\sigma = 0$. The same comment applies to the Gamma process case.

- (2) **Gamma process.** A Gamma process is a pure jump Lévy process with Lévy density $\rho(x) = c_+ x^{-1} e^{-\lambda_+ x} 1_{(0, \infty)}(x)$ (see Example 5.2). We denote this case by $\text{G}(c_+, \lambda_+)$. The increment distribution P_Δ is the Gamma distribution with shape parameter $c_+ \Delta$ and scale parameter $1/\lambda_+$. For the Gamma process case, we take $I = [0.25, 0.75]$.

We use the kernel function W whose Fourier transform φ_W is specified by (4.4), where we choose $b = 1$ and $c = 0.05$. We consider the following configurations for the sample size n and the time span Δ : $n = 5 \times 10^4$ or 10^5 , and $\Delta = 0.01$ or 0.005 . Here $n\Delta$ ranges from 250 to 1000.

Remark 6.1. From a theoretical point of view, we do not have to estimate σ^2 even when $\sigma > 0$ (see the discussion on Condition (vi) in Assumption 4.1). However, in case of $\sigma > 0$, we found that the estimate $\hat{\rho}$ with $\hat{\sigma}^2 = 0$ tends to be less precise near the origin than that with $\hat{\sigma}^2 = \hat{\sigma}_{TRV}^2$. So, from a practical point of view, we recommend to estimate σ^2 when implementing our methods.

6.2. Bandwidth selection. Now, we discuss bandwidth selection. We adapt an idea of [11] on bandwidth selection in density deconvolution. From a theoretical point of view, for our confidence bands to work, we have to choose bandwidths that are of smaller order than the optimal rate for estimation under the the sup-norm loss. At the same time, choosing a too small bandwidth

results in a too wide confidence band. Therefore, heuristically, we should choose a bandwidth “slightly” smaller than the optimal one that minimizes the L^∞ -distance $\|\widehat{\rho} - \rho\|_I$.

Let $\widehat{\rho}_h$ denote the spectral estimate with bandwidth h . Figure 3 depicts five realizations of the L^∞ -distance $\|\widehat{\rho}_h - \rho\|_I$ with different bandwidth values for BCN(0,1/2) with $\lambda = 4$ and G(0.2,1), both with $(n, \Delta) = (5 \times 10^4, 0.01)$. It is observed that as h increases, $\|\widehat{\rho}_h - \rho\|_I$ is sharply decreasing for $h < h^*$ (say), and for $h > h^*$, $\|\widehat{\rho}_h - \rho\|_I$ is slowly increasing. We aim to choose a bandwidth slightly smaller than h^* . Of course, the problem is that the value of $\|\widehat{\rho}_h - \rho\|_I$ is unknown to us. Now, Figure 4 depicts five realizations of the L^∞ -distance between the estimates of ρ with adjacent bandwidth values. To be precise, we prepare grids of bandwidths $h_1 < \dots < h_J$, and compute the L^∞ -distance $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$; Figure 4 depicts those values with $h = h_j$ ($j = 2, \dots, J$). It is observed that shape of $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ partly “mimics” that of $\|\widehat{\rho}_h - \rho\|_I$; in fact, $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ is sharply decreasing for $h_j < h^*$, but for $h_j > h^*$, $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ is almost flat. Our idea of bandwidth selection is as follows: starting from $j = 2$, choose the first j such that $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ is below $\kappa \times \min\{\|\widehat{\rho}_{h_k} - \widehat{\rho}_{h_{k-1}}\|_I : k = 2, \dots, J\}$ for some $\kappa > 1$; our choice of the bandwidth is $h = h_j$. Heuristically, this rule would choose a bandwidth “slightly” smaller than h^* (as long as the threshold value κ is reasonably chosen). Formally, we employ the following rule for bandwidth selection.

- (1) Set a pilot bandwidth $h^P = M\Delta^{1/2}$ for some $M > 1$, and make a list of candidate bandwidths $h_j = jh^P/J$ for $j = 1, \dots, J$.
- (2) Choose the smallest bandwidth h_j ($j \geq 2$) such that the adjacent L^∞ -distance $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ is smaller than $\kappa \times \min\{\|\widehat{\rho}_{h_k} - \widehat{\rho}_{h_{k-1}}\|_I : k = 2, \dots, J\}$ for some $\kappa > 1$

In this simulation study, we choose $M = 2, J = 20$, and $\kappa = 20$. In practice, it is also recommended to make use of visual information on how $\|\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}\|_I$ behaves as j increases when determining the bandwidth.

Remark 6.2. We have also examined a version of the bandwidth selection rule with $\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}}$ replaced by $x^2(\widehat{\rho}_{h_j} - \widehat{\rho}_{h_{j-1}})$, but found that the rule described above shows better performances in terms of coverage probabilities. So, we present simulation results with the above rule.

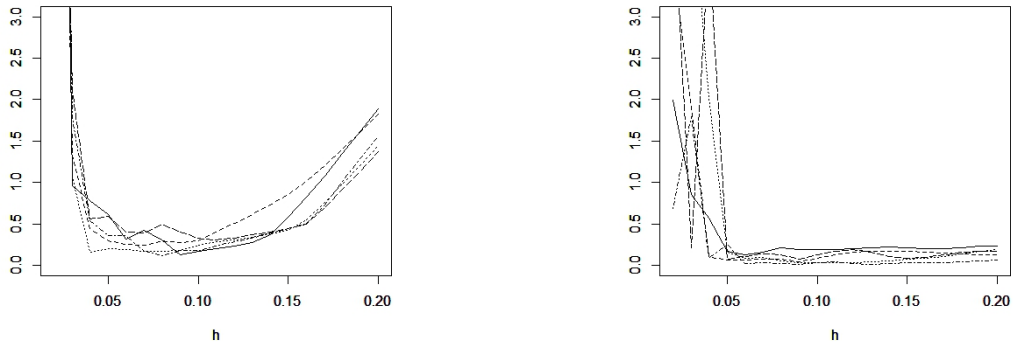


FIGURE 3. L^∞ -distance between the true Lévy density ρ and estimates $\widehat{\rho}$ for different bandwidth values. The left figure corresponds to BCN(0,1/2) with $\lambda = 4$, and the right figure corresponds to G(0.2,1), both with $(n, \Delta) = (5 \times 10^4, 0.01)$.

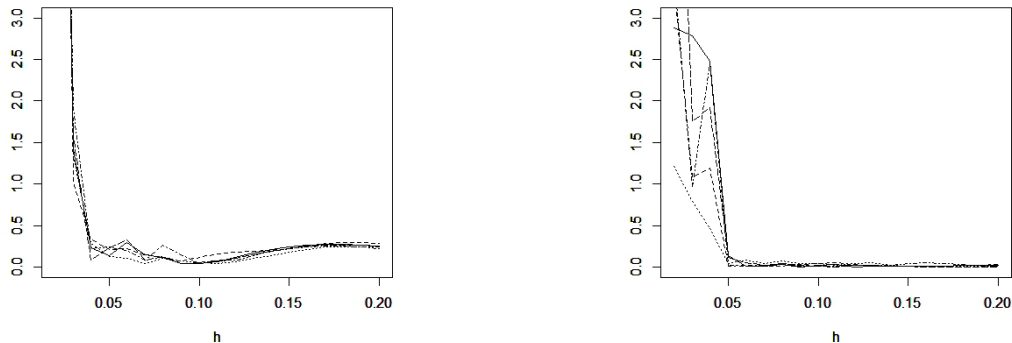


FIGURE 4. L^∞ -distance between the estimates of ρ with adjacent bandwidths for BCN(0,1/2) with $\lambda = 4$ (left) and G(0.2,1) (right).

6.3. Simulation results. In this simulation study, we focus on the multiplier bootstrap (MB) confidence band $\widehat{\mathcal{C}}_{1-\tau}^{MB}$. We present simulated coverage probabilities of the MB confidence band together with simulated values of the expected mean width of the band $E \left[\frac{1}{|I|} \int_I |\widehat{\mathcal{C}}_{1-\tau}^{MB}(x)| dx \right]$, where $|A|$ denotes the Lebesgue measure for a measurable set $A \subset \mathbb{R}$. The number of Monte Carlo repetitions is 250. To compute the critical value $\widehat{c}_n^{MB}(1-\tau)$, we generate 1,500 multiplier bootstrap replications for each run of the simulations.

Tables 1 and 2 present simulation results for models BCN(σ, v) and BCL(σ, v) with $\sigma \in \{0, 1\}$, $v = 1/2$, and G(0.2, 1) under $(n, \Delta) = (5 \times 10^4, 0.005), (10^5, 0.005), (5 \times 10^4, 0.01),$ and $(10^5, 0.01)$. Comparing BCN(0, 1/2) with BCL(0, 1/2) and G(0.2, 1), we find that BCN(0, 1/2) is apt to give more accurate coverage probabilities. This is partly due to the smoothness of Lévy densities. Since the normal density is smoother around the origin than those of Laplace and Gamma-Lévy densities, the estimate $\widehat{\rho}$ for BCN tends to be less biased than that for other cases. Figure 5 depicts 90% MB confidence bands for BCN(0, 1/2) (left), BCL(0, 1/2) (center), both with $\lambda = 10$, and G(0.2, 1) (right) with $(n, \Delta) = (5 \times 10^4, 0.01)$ (top row) and $(10^5, 0.01)$ (bottom row), based on one realization for each model. Figure 6 depicts 90% MB confidence bands for BCN(1, 1/2) and BCL(1, 1/2), both with $\lambda = 10$. As seen from Figures 5 and 6, the width of the 90% MB confidence band tends to increase near the origin when the Brownian component is present (i.e., $\sigma = 1$). This partly comes from the difficulty of distinguishing small jumps from fluctuations due to the Brownian component.

Overall, the simulated coverage probabilities are reasonably close to the nominal coverage probabilities, although in some cases there are rooms for improvement. We also find that for every case, the expected mean width tends to decrease as $n\Delta$ increases, which is consistent with our theory. Notably, for BCN and BCL, the MB confidence bands exhibit similar performance for either case where the Brownian component is absent ($\sigma = 0$) or present ($\sigma = 1$).

| Cov. Prob. ($1 - \tau$) | Δ | n | Model | | | | G(0.2,1) | |
|------------------------------|----------|-----------------|------------------|------------------|------------------|------------------|------------------|------------------|
| | | | BCN(0,1/2) | BCN(0,1/2) | BCL(0,1/2) | BCL(0,1/2) | | |
| | | | $\lambda = 4$ | $\lambda = 10$ | $\lambda = 4$ | $\lambda = 10$ | | |
| 0.90 | 0.005 | 5×10^4 | 0.816 (1.150) | 0.824 (1.867) | 0.812 (0.818) | 0.824 (1.294) | 0.812 (0.317) | |
| | | 10^5 | 0.828 (0.816) | 0.836 (1.311) | 0.820 (0.573) | 0.808 (0.908) | 0.812 (0.213) | |
| | 0.01 | 5×10^4 | 0.824 (0.787) | 0.840 (1.285) | 0.820 (0.560) | 0.816 (0.922) | 0.820 (0.195) | |
| | | 10^5 | 0.868 (0.545) | 0.856 (0.905) | 0.824 (0.399) | 0.796 (0.659) | 0.816 (0.131) | |
| | 0.95 | 0.005 | 5×10^4 | 0.908 (1.276) | 0.912 (2.071) | 0.908 (0.919) | 0.916 (1.453) | 0.908 (0.364) |
| | | | 10^5 | 0.912 (0.908) | 0.924 (1.455) | 0.908 (0.643) | 0.904 (1.019) | 0.920 (0.245) |
| 0.01 | | 5×10^4 | 0.916 (0.876) | 0.928 (1.428) | 0.912 (0.631) | 0.916 (1.037) | 0.912 (0.226) | |
| | | 10^5 | 0.932 (0.607) | 0.936 (1.004) | 0.920 (0.449) | 0.876 (0.740) | 0.904 (0.153) | |
| 0.99 | | 0.005 | 5×10^4 | 0.972 (1.532) | 0.976 (2.441) | 0.968 (1.110) | 0.984 (1.742) | 0.964 (0.454) |
| | | | 10^5 | 0.988 (1.090) | 0.984 (1.712) | 0.980 (0.771) | 0.976 (1.235) | 0.984 (0.301) |
| | 0.01 | 5×10^4 | 0.972 (1.044) | 0.988 (1.695) | 0.984 (0.767) | 0.988 (1.265) | 0.980 (0.285) | |
| | | 10^5 | 0.984 (0.742) | 0.992 (1.184) | 0.992 (0.540) | 0.964 (0.892) | 0.988 (0.193) | |

TABLE 1. Empirical coverage probabilities of the MB confidence bands for BCN(0,1/2) and BCL(0,1/2) on $I = [-0.75, -0.25] \cup [0.25, 0.75]$, and G(0.2,1) on $I = [0.25, 0.75]$, based on 250 Monte Carlo repetitions. Inside the parentheses are values of the expected mean width.

| Cov. Prob. ($1 - \tau$) | Model | | | | | | |
|------------------------------|----------|-----------------|------------------|------------------|------------------|------------------|------------------|
| | Δ | n | BCN(1,1/2) | BCN(1,1/2) | BCL(1,1/2) | BCL(1,1/2) | |
| | | | $\lambda = 4$ | $\lambda = 10$ | $\lambda = 4$ | $\lambda = 10$ | |
| 0.90 | 0.005 | 5×10^4 | 0.804 (1.447) | 0.804 (2.425) | 0.832 (1.002) | 0.828 (1.536) | |
| | | 10^5 | 0.808 (1.050) | 0.796 (1.756) | 0.820 (0.699) | 0.816 (1.070) | |
| | 0.01 | 5×10^4 | 0.812 (1.113) | 0.808 (1.900) | 0.824 (0.870) | 0.804 (1.303) | |
| | | 10^5 | 0.824 (0.811) | 0.804 (1.409) | 0.816 (0.590) | 0.792 (0.946) | |
| | 0.95 | 0.005 | 5×10^4 | 0.904 (1.606) | 0.904 (2.692) | 0.912 (1.119) | 0.908 (1.720) |
| | | | 10^5 | 0.908 (1.165) | 0.892 (1.945) | 0.916 (0.784) | 0.916 (1.199) |
| 0.01 | | 5×10^4 | 0.912 (1.241) | 0.908 (2.116) | 0.920 (0.975) | 0.904 (1.460) | |
| | | 10^5 | 0.916 (0.901) | 0.896 (1.568) | 0.916 (0.661) | 0.896 (1.056) | |
| 0.99 | | 0.005 | 5×10^4 | 0.956 (1.840) | 0.968 (3.113) | 0.956 (1.367) | 0.988 (2.067) |
| | | | 10^5 | 0.960 (1.392) | 0.972 (2.301) | 0.972 (0.947) | 0.980 (1.460) |
| | 0.01 | 5×10^4 | 0.972 (1.478) | 0.976 (2.465) | 0.984 (1.149) | 0.976 (1.743) | |
| | | 10^5 | 0.976 (1.059) | 0.964 (1.812) | 0.964 (0.798) | 0.968 (1.273) | |

TABLE 2. Empirical coverage probabilities of the MB confidence bands for BCN(1,1/2) and BCL(1,1/2) on $I = [-0.75, -0.25] \cup [0.25, 0.75]$, based on 250 Monte Carlo repetitions. Inside the parentheses are values of the expected mean width.

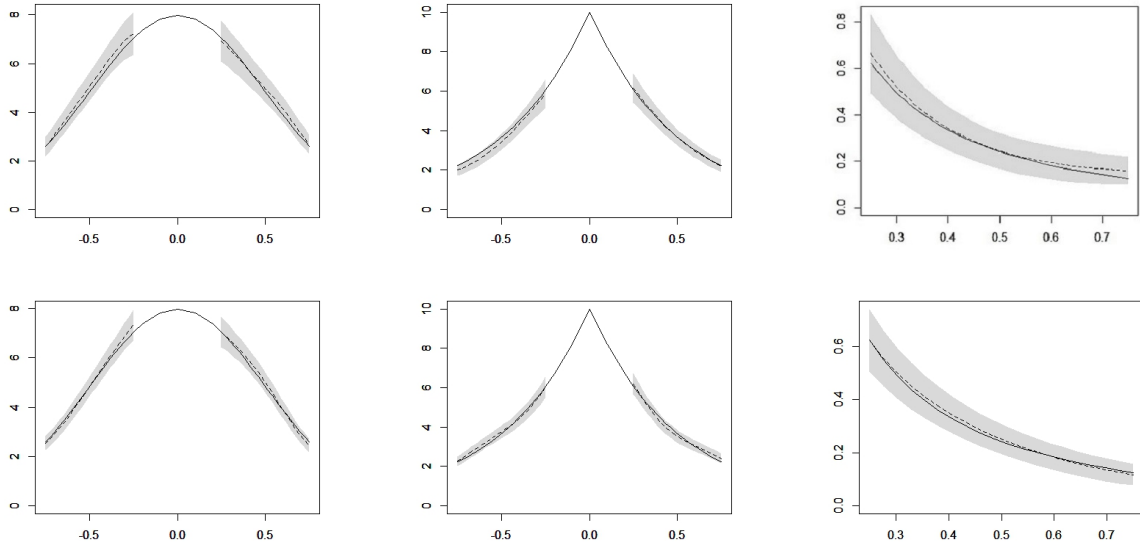


FIGURE 5. Estimates of the Lévy densities (dashed lines) for $\text{BCN}(0,1/2)$ (left), $\text{BCL}(0,1/2)$ (center), and $\text{G}(0.2,1)$ (right), together with 90% MB confidence bands (gray regions). The solid lines correspond to the true density functions. $(n, \Delta) = (5 \times 10^4, 0.01)$ (top row) and $(n, \Delta) = (10^5, 0.01)$ (bottom row).

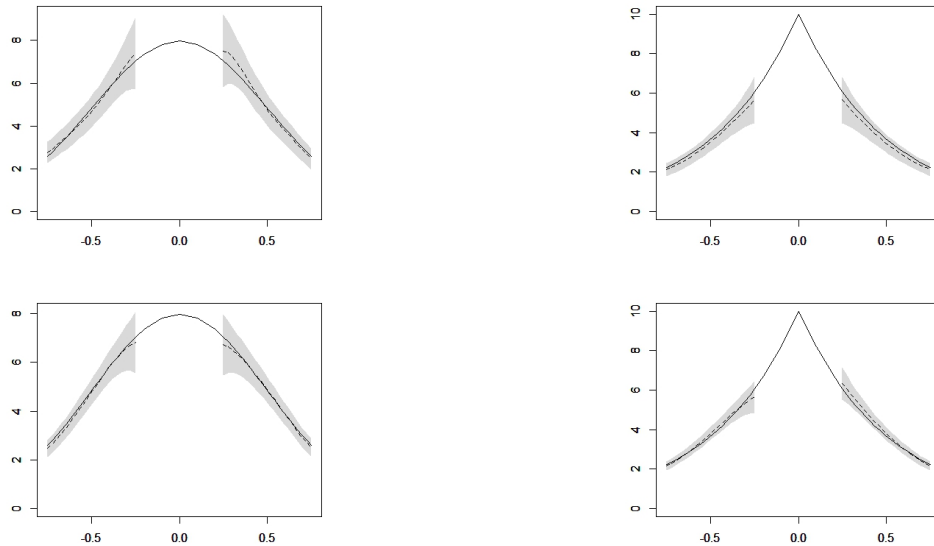


FIGURE 6. Estimates of the Lévy densities (dashed lines) for $\text{BCN}(1,1/2)$ (left) and $\text{BCL}(1,1/2)$ (right), together with 90% MB confidence bands (gray regions). The solid lines correspond to the true density functions. $(n, \Delta) = (5 \times 10^4, 0.01)$ (top row) and $(n, \Delta) = (10^5, 0.01)$ (bottom row).

7. PROOFS OF THEOREMS 4.1 AND 4.2

In what follows, we always assume Assumption 4.1. The proofs rely on modern empirical process theory. For a probability measure Q on a measurable space (S, \mathcal{S}) and a class of measurable functions \mathcal{F} on S such that $\mathcal{F} \subset L^2(Q)$, let $N(\mathcal{F}, \|\cdot\|_{Q,2}, \varepsilon)$ denote the ε -covering number for \mathcal{F} with respect to the $L^2(Q)$ -seminorm $\|\cdot\|_{Q,2}$. See Section 2.1 in [63] for details. Let $\stackrel{d}{=}$ denote the equality in distribution.

7.1. Auxiliary lemmas. We begin with proving some auxiliary lemmas that will be used to prove Theorems 4.1 and 4.2. We will freely use the following moment estimates for $L_\Delta = Y_{n,1}$.

Lemma 7.1. *We have*

$$\mathbb{E}[L_\Delta^2] = \Delta \left(\sigma^2 + \int_{\mathbb{R}} x^2 \rho(x) dx \right) + \Delta^2 \gamma_c^2 \lesssim \Delta \quad \text{and} \quad \mathbb{E}[L_\Delta^4] = \Delta \int_{\mathbb{R}} x^4 \rho(x) dx + o(\Delta) \lesssim \Delta.$$

Proof. This follows from the observations that $\mathbb{E}[L_\Delta^2] = -\varphi_\Delta''(0)$ and $\mathbb{E}[L_\Delta^4] = \varphi_\Delta^{(4)}(0)$. \square

Lemma 7.2. *We have $\inf_{|u| \leq h^{-1}} |\varphi_\Delta(u)| \gtrsim 1$.*

Proof. Recall that $\varphi_\Delta(u) = e^{\Delta\psi(u)}$. From Taylor's theorem, $|e^{iux} - 1 - iux| \leq \frac{u^2 x^2}{2}$ for any $u, x \in \mathbb{R}$, so that $|\psi(u)| \leq \frac{\sigma^2 u^2}{2} + |\gamma_c| |u| + \frac{u^2}{2} \int_{\mathbb{R}} x^2 \rho(x) dx \lesssim h^{-2} \lesssim \Delta^{-1}$ uniformly in $|u| \leq h^{-1}$. Therefore, we conclude that

$$\inf_{|u| \leq h^{-1}} |\varphi_\Delta(u)| \geq e^{-\Delta \sup_{|u| \leq h^{-1}} |\psi(u)|} = e^{-O(1)} \gtrsim 1.$$

This completes the proof. \square

Lemma 7.3. *We have $\|(1 + x^2 + h^2 x^4)(|K_n| \vee |K'_n|)\|_{\mathbb{R}} \lesssim 1$.*

Proof. From the previous lemma, it is not difficult to verify that $\|K_n\|_{\mathbb{R}} \lesssim 1$ and $\|K'_n\|_{\mathbb{R}} \lesssim 1$. By changes of variables, observe that

$$K_n(x) = \frac{1}{2\pi x^2} \int_{\mathbb{R}} e^{-iux} \left(\frac{\varphi_W(u)}{\varphi_\Delta(u/h)} \right)'' du = \frac{-1}{2\pi x^4} \int_{\mathbb{R}} e^{-iux} \left(\frac{\varphi_W(u)}{\varphi_\Delta(u/h)} \right)^{(4)} du.$$

Since φ_W is supported in $[-1, 1]$, to show that $\|(x^2 + h^2 x^4)K_n\|_{\mathbb{R}} \lesssim 1$, it suffices to verify that

$$\left\| \left(\frac{\varphi_W(\cdot)}{\varphi_\Delta(\cdot/h)} \right)'' \right\|_{[-1,1]} \lesssim 1 \quad \text{and} \quad \left\| \left(\frac{\varphi_W(\cdot)}{\varphi_\Delta(\cdot/h)} \right)^{(4)} \right\|_{[-1,1]} \lesssim h^{-2}.$$

To see this, observe that

$$\begin{aligned} \psi'(u) &= -\sigma^2 u + i\gamma_c + \int_{\mathbb{R}} (e^{iux} - 1)x\rho(x)dx, \quad \psi''(u) = -\sigma^2 - \int_{\mathbb{R}} e^{iux} x^2 \rho(x)dx, \\ \psi'''(u) &= -i \int_{\mathbb{R}} e^{iux} x^3 \rho(x)dx, \quad \psi^{(4)}(u) = \int_{\mathbb{R}} e^{iux} x^4 \rho(x)dx, \\ \varphi'_\Delta &= \Delta \psi' \varphi_\Delta, \quad \varphi''_\Delta = \Delta \{\psi'' + \Delta(\psi')^2\} \varphi_\Delta, \\ \varphi'''_\Delta &= \Delta \{\psi''' + 3\Delta \psi'' \psi' + \Delta^2 (\psi')^3\} \varphi_\Delta(u), \\ \varphi^{(4)}_\Delta &= \Delta \left[\psi^{(4)} + \Delta \{4\psi''' \psi' + 3(\psi'')^2\} + 6\Delta^2 \psi'' (\psi')^2 + \Delta^3 (\psi')^4 \right] \varphi_\Delta. \end{aligned}$$

This yields that

$$\begin{aligned} \left\| \frac{\varphi'_\Delta}{\varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} &\lesssim \Delta h^{-1}, \quad \left\| \frac{\varphi''_\Delta}{\varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} \lesssim \Delta \{1 + \Delta h^{-2}\} \lesssim \Delta, \\ \left\| \frac{\varphi'''_\Delta}{\varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} &\lesssim \Delta \{1 + \Delta h^{-1} + \Delta^2 h^{-3}\} \lesssim \Delta, \\ \left\| \frac{\varphi^{(4)}_\Delta}{\varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} &\lesssim \Delta \{1 + \Delta(h^{-1} + 1) + \Delta^2 h^{-2} + \Delta^3 h^{-4}\} \lesssim \Delta, \end{aligned}$$

where we have used that $h \gtrsim \Delta^{1/2}$. Next, observe the following identities

$$\begin{aligned} \left(\frac{f}{g} \right)'' &= \frac{f''}{g} - 2 \frac{f' g'}{g^2} + \frac{f}{g} \left\{ -\frac{g''}{g} + 2 \left(\frac{g'}{g} \right)^2 \right\}, \\ \left(\frac{f}{g} \right)^{(4)} &= \frac{f^{(4)}}{g} - 4 \frac{f''' g'}{g^2} + 6 \frac{f''}{g} \left\{ -\frac{g''}{g} + 2 \left(\frac{g'}{g} \right)^2 \right\} + 4 \frac{f'}{g} \left\{ -\frac{g'''}{g} + 6 \frac{g' g''}{g^2} - 6 \left(\frac{g'}{g} \right)^3 \right\} \\ &\quad + \frac{f}{g} \left\{ -\frac{g^{(4)}}{g} + 8 \frac{g' g'''}{g^2} + 6 \left(\frac{g''}{g} \right)^2 - 36 \left(\frac{g'}{g} \right)^2 \frac{g''}{g} + 24 \left(\frac{g'}{g} \right)^4 \right\}. \end{aligned}$$

The second identity follows from the following (straightforward but tedious) calculations:

$$\begin{aligned} \left(\frac{f}{g} \right)^{(4)} &= \frac{f^{(4)}}{g} + 4f''' \left(\frac{1}{g} \right)' + 6f'' \left(\frac{1}{g} \right)'' + 4f' \left(\frac{1}{g} \right)''' + f \left(\frac{1}{g} \right)^{(4)}, \\ \left(\frac{1}{g} \right)' &= -\frac{g'}{g^2} = -\frac{1}{g} \frac{g'}{g}, \\ \left(\frac{1}{g} \right)'' &= -\frac{g''}{g^2} + 2 \frac{(g')^2}{g^3} = \frac{1}{g} \left\{ -\frac{g''}{g} + 2 \left(\frac{g'}{g} \right)^2 \right\}, \\ \left(\frac{1}{g} \right)''' &= -\frac{g'''}{g^2} + 6 \frac{g' g''}{g^3} - 6 \frac{(g')^3}{g^4} = \frac{1}{g} \left\{ -\frac{g'''}{g} + 6 \frac{g' g''}{g^2} - 6 \left(\frac{g'}{g} \right)^3 \right\}, \\ \left(\frac{1}{g} \right)^{(4)} &= -\frac{g^{(4)}}{g^2} + 8 \frac{g' g'''}{g^3} + 6 \frac{(g'')^2}{g^3} - 36 \frac{(g')^2 g''}{g^4} + 24 \frac{(g')^4}{g^5} \\ &= \frac{1}{g} \left\{ -\frac{g^{(4)}}{g} + 8 \frac{g' g'''}{g^2} + 6 \left(\frac{g''}{g} \right)^2 - 36 \left(\frac{g'}{g} \right)^2 \frac{g''}{g} + 24 \left(\frac{g'}{g} \right)^4 \right\}. \end{aligned}$$

Now, noting that $(\varphi_\Delta(u/h))^{(k)} = h^{-k} \varphi_\Delta^{(k)}(u/h)$ for $k = 1, 2, 3, 4$, we conclude that

$$\begin{aligned} \left\| \left(\frac{\varphi_W(\cdot)}{\varphi_\Delta(\cdot/h)} \right)'' \right\|_{[-1,1]} &\lesssim 1 + \Delta h^{-2} + \{\Delta h^{-2} + \Delta^2 h^{-4}\} \lesssim 1, \\ \left\| \left(\frac{\varphi_W(\cdot)}{\varphi_\Delta(\cdot/h)} \right)^{(4)} \right\|_{[-1,1]} &\lesssim 1 + \Delta h^{-2} + \{\Delta h^{-2} + \Delta^2 h^{-4}\} + \{\Delta h^{-3} + \Delta^2 h^{-4} + \Delta^3 h^{-6}\} \\ &\quad + \{\Delta h^{-4} + \Delta^2 h^{-5} + \Delta^2 h^{-4} + \Delta^3 h^{-6} + \Delta^4 h^{-8}\} \\ &\lesssim 1 + \Delta h^{-4} \lesssim h^{-2}. \end{aligned}$$

Likewise, we have that $\|(x^2 + h^2 x^4)K'_n\|_{\mathbb{R}} \lesssim 1$. This completes the proof. \square

Lemma 7.4. *For $k = 0, 1, 2$, we have that $\|(\widehat{\varphi}_\Delta - \varphi_\Delta)^{(k)}\|_{[-h^{-1}, h^{-1}]} = O_P\{n^{-1/2} \Delta^{(k \wedge 1)/2} \log h^{-1}\}$.*

Proof. This follows from Theorem 1 in [44], which shows that for the weight function $w(u) = (\log(e + |u|))^{-1}$,

$$C_k := \sup_n \mathbb{E}[\|\sqrt{n} \Delta^{-(k \wedge 1)/2} (\widehat{\varphi}_\Delta - \varphi_\Delta)^{(k)} w\|_{\mathbb{R}}] < \infty$$

for $k = 0, 1, 2$ under our assumption. Since

$$\|\sqrt{n} \Delta^{-(k \wedge 1)/2} (\widehat{\varphi}_\Delta - \varphi_\Delta)^{(k)} w\|_{\mathbb{R}} \geq \sqrt{n} \Delta^{-(k \wedge 1)/2} \|(\widehat{\varphi}_\Delta - \varphi_\Delta)^{(k)}\|_{[-h^{-1}, h^{-1}]} \inf_{|u| \leq h^{-1}} w(u),$$

we conclude that

$$\mathbb{E}[\|(\widehat{\varphi}_\Delta - \varphi_\Delta)^{(k)}\|_{[-h^{-1}, h^{-1}]}] \leq \frac{C_k \Delta^{(k \wedge 1)/2}}{\sqrt{n} \inf_{|u| \leq h^{-1}} w(u)} \lesssim n^{-1/2} \Delta^{(k \wedge 1)/2} \log h^{-1}.$$

The desired result follows from Markov's inequality. \square

Lemmas 7.2 and 7.4 imply that

$$\inf_{|u| \leq h^{-1}} |\widehat{\varphi}_\Delta(u)| \geq \inf_{|u| \leq h^{-1}} |\varphi_\Delta(u)| - o_P(1) \gtrsim 1 - o_P(1),$$

so that with probability approaching one, $\inf_{|u| \leq h^{-1}} |\widehat{\varphi}_\Delta(u)| > 0$. Hence, with probability approaching one, $\widehat{\psi} := \Delta^{-1} \log \widehat{\varphi}_\Delta$ is well-defined on $[-h^{-1}, h^{-1}]$ as the distinguished logarithm [19, Theorem 7.6.2].

Lemma 7.5. *We have*

$$\left\| (\widehat{\psi} - \psi)'' - \frac{(\widehat{\varphi}_\Delta - \varphi_\Delta)''}{\Delta \varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} = o_P\{h(n \Delta h \log n)^{-1/2}\}.$$

Proof. The lemma essentially follows from the proof of Proposition 7 in [56]. For the sake of completeness, we provide a proof of the lemma. Rewrite $(\widehat{\psi} - \psi)''$ as $(\widehat{\psi} - \psi)'' = \Delta^{-1} (\log(\widehat{\varphi}_\Delta / \varphi_\Delta))''$. Let $F(y) = \log(1 + y)$, $\eta = (\widehat{\varphi}_\Delta - \varphi_\Delta) / \varphi_\Delta$, and observe that for any $|u| \leq h^{-1}$,

$$(F \circ \eta)''(u) = F'(\eta(u)) \eta''(u) + F''(\eta(u)) \eta'(u)^2 = \eta''(u) + F''(\theta \eta(u)) \eta(u) \eta''(u) + F''(\eta(u)) \eta'(u)^2$$

for some $\theta \in [0, 1]$. Since F'' is bounded in a neighborhood of the origin and $\|\eta\|_{[-h^{-1}, h^{-1}]} = o_P(1)$ (which follows from Lemmas 7.2 and 7.4), we have that

$$\begin{aligned} & \left\| (\log(\widehat{\varphi}_\Delta / \varphi_\Delta))'' - \left(\frac{\widehat{\varphi}_\Delta - \varphi_\Delta}{\varphi_\Delta} \right)'' \right\|_{[-h^{-1}, h^{-1}]} \\ &= O_P\left(\|\eta\|_{[-h^{-1}, h^{-1}]} \|\eta''\|_{[-h^{-1}, h^{-1}]} + \|\eta'\|_{[-h^{-1}, h^{-1}]}^2 \right). \end{aligned}$$

Next, we shall bound $\|\eta^{(k)}\|_{[-h^{-1}, h^{-1}]}$ for $k = 0, 1, 2$. Since $\|\varphi'_\Delta / \varphi_\Delta\|_{[-h^{-1}, h^{-1}]} \lesssim \Delta h^{-1} \lesssim \Delta^{1/2}$ and $\|\varphi''_\Delta / \varphi_\Delta\|_{[-h^{-1}, h^{-1}]} \lesssim \Delta$, we have that

$$\|(1/\varphi_\Delta)'\|_{[-h^{-1}, h^{-1}]} = \left\| \frac{\varphi'_\Delta}{\varphi_\Delta^2} \right\|_{[-h^{-1}, h^{-1}]} \lesssim \Delta^{1/2}, \quad (7.1)$$

$$\|(1/\varphi_\Delta)''\|_{[-h^{-1}, h^{-1}]} = \left\| \frac{\varphi''_\Delta \varphi_\Delta - 2(\varphi'_\Delta)^2}{\varphi_\Delta^3} \right\|_{[-h^{-1}, h^{-1}]} \lesssim \Delta. \quad (7.2)$$

In view of the identities

$$\begin{aligned}\eta' &= (1/\varphi_\Delta)'(\widehat{\varphi}_\Delta - \varphi_\Delta) + (1/\varphi_\Delta)(\widehat{\varphi}_\Delta - \varphi_\Delta)', \\ \eta'' &= (1/\varphi_\Delta)''(\widehat{\varphi}_\Delta - \varphi_\Delta) + 2(1/\varphi_\Delta)'(\widehat{\varphi}_\Delta - \varphi_\Delta)' + (1/\varphi_\Delta)(\widehat{\varphi}_\Delta - \varphi_\Delta)'',\end{aligned}$$

we conclude from Lemma 7.4 that $\|\eta^{(k)}\|_{[-h^{-1}, h^{-1}]} = O_P(n^{-1/2}\Delta^{(k\wedge 1)/2}\log n)$ for $k = 0, 1, 2$, which yields that

$$\|\eta\|_{[-h^{-1}, h^{-1}]} \|\eta''\|_{[-h^{-1}, h^{-1}]} \vee \|\eta'\|_{[-h^{-1}, h^{-1}]}^2 = O_P\{n^{-1}\Delta^{1/2}(\log n)^2\}.$$

Finally, observe that

$$(1/\varphi_\Delta)(\widehat{\varphi}_\Delta - \varphi_\Delta)'' = \eta'' - (1/\varphi_\Delta)''(\widehat{\varphi}_\Delta - \varphi_\Delta) - 2(1/\varphi_\Delta)'(\widehat{\varphi}_\Delta - \varphi_\Delta)'.$$

From the bounds (7.1) and (7.2), together with Lemma 7.4, we have that

$$\|(1/\varphi_\Delta)''(\widehat{\varphi}_\Delta - \varphi_\Delta)\|_{[-h^{-1}, h^{-1}]} \vee \|(1/\varphi_\Delta)'(\widehat{\varphi}_\Delta - \varphi_\Delta)'\|_{[-h^{-1}, h^{-1}]} = O_P(n^{-1/2}\Delta\log n).$$

Taking these together, we conclude that

$$\left\| (\log(\widehat{\varphi}_\Delta/\varphi_\Delta))'' - \frac{(\widehat{\varphi}_\Delta - \varphi_\Delta)''}{\varphi_\Delta} \right\|_{[-h^{-1}, h^{-1}]} = O_P\{n^{-1/2}\Delta\log n + n^{-1}\Delta^{1/2}(\log n)^2\}. \quad (7.3)$$

We shall verify that the right hand side is $o_P\{\Delta h(n\Delta h\log n)^{-1/2}\}$. Observe that

$$\frac{n^{-1/2}\Delta\log n + n^{-1}\Delta^{1/2}(\log n)^2}{\Delta h(n\Delta h\log n)^{-1/2}} = \Delta^{1/2}h^{-1/2}(\log n)^{3/2} + n^{-1/2}h^{-1/2}(\log n)^{5/2}.$$

Since $\sqrt{nh} \gg n^\delta \Delta^{-1/2}$, we have that $n^{-1/2}h^{-1/2}(\log n)^{5/2} \ll n^{-\delta}\Delta^{1/2}(\log n)^{5/2} \ll 1$. On the other hand, as $h \gtrsim \Delta^{1/2}$, $\Delta^{1/2}h^{-1/2}(\log n)^{3/2} \lesssim \Delta^{1/4}(\log n)^{3/2}$. Since $1 \gg h^r \sqrt{n\Delta h \log n} \gtrsim \Delta^{r/2+3/4} \sqrt{n \log n}$, we have that $\Delta \ll (n \log n)^{-1/(r+3/2)}$, which implies that $\Delta^{1/4}(\log n)^{3/2} \ll 1$. This completes the proof. \square

Lemma 7.6. *Recall that $s_n^2(x) = \text{Var}(Y_{n,1}^2 K_n((x - Y_{n,1})/h))$. Then $\inf_{x \in I} s_n^2(x) \gtrsim \Delta h$ for sufficiently large n .*

Proof. Since $\|K_n\|_{\mathbb{R}} \lesssim 1$, we have that $(\mathbb{E}[Y_{n,1}^2 K_n((x - Y_{n,1})/h)])^2 \leq \|K_n\|_{\mathbb{R}}^2 (\mathbb{E}[Y_{n,1}^2])^2 \lesssim \Delta^2 \ll \Delta h$. Next, observe that $(x+y)^4 \geq x^4/16 - y^4$ for any $x, y \in \mathbb{R}$. Using this inequality and recalling that $Y_{n,1} - b\Delta$ has distribution $P_{\Delta, b}$ with $P_{\Delta, b}(dy) = g_{\Delta, b}(y)dy$, we have that

$$\begin{aligned}\mathbb{E}[Y_{n,1}^4 K_n^2((x - Y_{n,1})/h)] &= \mathbb{E}[(Y_{n,1} - b\Delta + b\Delta)^4 K_n^2((x - (Y_{n,1} - b\Delta) - b\Delta)/h)] \\ &\geq \frac{1}{16} \int_{\mathbb{R}} K_n^2((x - b\Delta - y)/h) g_{\Delta, b}(y) dy - b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2 \\ &= \frac{h}{16} \int_{\mathbb{R}} K_n^2(y) g_{\Delta, b}(x - b\Delta - yh) dy - b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2.\end{aligned}$$

Since $\Delta^4 \ll \Delta h$, it suffices to verify that

$$\inf_{x \in I^{[b\Delta]}} \int_{\mathbb{R}} K_n^2(y) g_{\Delta, b}(x - yh) dy \gtrsim \Delta$$

for sufficiently large n . Since $|\varphi_\Delta| \leq 1$ and by Plancherel's theorem, we have that

$$\int_{\mathbb{R}} K_n^2(y) dy = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\varphi_W(u)}{\varphi_\Delta(u/h)} \right|^2 du \geq \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_W(u)|^2 du =: \underline{c}.$$

From Lemma 7.3, we have that $\|(1 + y^2)K_n\|_{\mathbb{R}} \lesssim 1$, so that for $m > 0$,

$$\int_{[-m, m]^c} K_n^2(y) dy \lesssim \int_{[-m, m]^c} y^{-4} dy$$

up to a constant independent of (n, m) . The right hand side is approaching zero as $m \rightarrow \infty$, so that by taking m sufficiently large, we have that

$$\int_{-m}^m K_n^2(y) dy \geq \int_{\mathbb{R}} K_n^2(y) dy - \frac{c}{2} \geq \frac{c}{2}$$

for all n . Hence, for sufficiently large n such that $|b|\Delta + mh \leq \varepsilon_0$,

$$\inf_{x \in I^{|b|\Delta}} \int_{\mathbb{R}} K_n^2(y) g_{\Delta, b}(x - yh) dy \geq \inf_{x \in I^{|b|\Delta}} \int_{-m}^m K_n^2(y) g_{\Delta, b}(x - yh) dy \geq \frac{c}{2} \inf_{x \in I^{\varepsilon_0}} g_{\Delta, b}(x) \gtrsim \Delta$$

by Condition (ii) in Assumption 4.1. This completes the proof. \square

Consider the function class

$$\mathcal{F}_n = \left\{ y \mapsto \frac{y^2}{s_n(x)} K_n \left(\frac{x - y}{h} \right) : x \in I \right\}.$$

Observe that

$$y^2 K_n \left(\frac{x - y}{h} \right) = h^2 \left(\frac{x - y}{h} \right)^2 K_n \left(\frac{x - y}{h} \right) - 2xh \left(\frac{x - y}{h} \right) K_n \left(\frac{x - y}{h} \right) + x^2 K_n \left(\frac{x - y}{h} \right).$$

Since $\|(1 + y^2)K_n\|_{\mathbb{R}} \lesssim 1$ by Lemma 7.3 and I is compact, each of the three terms on the right hand side is bounded (as a function of y) uniformly in n and $x \in I$. Choose constants $D_1, D_2 > 0$ independent of n such that $\|(1 + y^2)K_n\|_{\mathbb{R}} \leq D_1$ and $\|1/s_n\|_I \leq D_2/\sqrt{\Delta h}$ (cf. Lemma 7.6). Without loss of generality, we may assume that $h \leq 1$. Then functions in \mathcal{F}_n are bounded by

$$(1 + 2\|x\|_I + \|x^2\|_I) D_1 D_2 / \sqrt{\Delta h} \lesssim 1 / \sqrt{\Delta h}.$$

The next lemma provides a bound on the uniform covering number for the function class \mathcal{F}_n .

Lemma 7.7. *There exist constants $A, v > 0$ independent of n such that*

$$\sup_Q N(\mathcal{F}_n, \|\cdot\|_{Q, 2}, \varepsilon / \sqrt{\Delta h}) \leq (A/\varepsilon)^v, \quad 0 < \forall \varepsilon \leq 1, \quad (7.4)$$

where \sup_Q is taken over all finitely discrete distributions on \mathbb{R} .

The proof of this lemma relies on the following lemma.

Lemma 7.8 ([34], Lemma 3.2.16). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of bounded variation, i.e.,*

$$\|f\|_{\text{TV}} := \sup \left\{ \sum_{j=1}^N |f(x_j) - f(x_{j-1})| : -\infty < x_0 < \dots < x_N < \infty, N = 1, 2, \dots \right\} < \infty,$$

and consider the function class $\mathcal{F} = \{x \mapsto f(ax + b) : a, b \in \mathbb{R}\}$. Then there exist universal constants $A, v > 0$ such that

$$\sup_Q N(\mathcal{F}, \|\cdot\|_{Q, 2}, \varepsilon \|f\|_{\text{TV}}) \leq (A/\varepsilon)^v, \quad 0 < \forall \varepsilon \leq 1,$$

where \sup_Q is taken over all finitely discrete distributions on \mathbb{R} .

Proof of Lemma 7.7. Consider the auxiliary function classes

$$\mathcal{G}_{n,\ell} = \left\{ y \mapsto h^\ell (ay + b)^\ell K_n(ay + b) : a, b \in \mathbb{R} \right\}, \quad \ell = 0, 1, 2.$$

Since $\mathcal{F}_n \subset \{(g_2 - 2xg_1 + x^2g_0)/s_n(x) : g_\ell \in \mathcal{G}_{n,\ell}, \ell = 0, 1, 2; x \in I\}$, I is compact, and $\|1/s_n\|_I \lesssim 1/\sqrt{\Delta h}$, the desired conclusion follows by verifying that there exist constants $A_1, v_1 > 0$ independent of n such that

$$\sup_Q N(\mathcal{G}_{n,\ell}, \|\cdot\|_{Q,2}, \varepsilon) \leq (A_1/\varepsilon)^{v_1}, \quad 0 < \forall \varepsilon \leq 1$$

for $\ell = 0, 1, 2$. To this end, in view of Lemma 7.8, it suffices to verify that, for $K_{n,\ell}(y) := y^\ell h^\ell K_n(y)$, $\ell = 0, 1, 2$, the total variations of $K_{n,\ell}$ are bounded in n , i.e.,

$$\|K_{n,\ell}\|_{\text{TV}} = \int_{\mathbb{R}} |K'_{n,\ell}(y)| dy \lesssim 1.$$

This follows from observations that $K'_{n,1}(y) = h(K_n(y) + yK'_n(y))$, $K'_{n,2}(y) = h^2(2yK_n(y) + y^2K'_n(y))$, and $\|(1 + y^2 + h^2y^4)(|K_n| \vee |K'_n|)\|_{\mathbb{R}} \lesssim 1$ by Lemma 7.3. \square

Lemma 7.9. *Let $\rho_{\sharp}(x) = x^2\rho(x)$. Then $\|\rho_{\sharp} * (h^{-1}W(\cdot/h)) - \rho_{\sharp}\|_{\mathbb{R}} \lesssim h^r$.*

Proof. Observe that by a change of variables, $[\rho_{\sharp} * (h^{-1}W(\cdot/h))](x) - \rho_{\sharp}(x) = \int_{\mathbb{R}} \{\rho_{\sharp}(x - yh) - \rho_{\sharp}(x)\} W(y) dy$. If $p \geq 1$, then by Taylor's theorem, for any $x, y \in \mathbb{R}$,

$$\rho_{\sharp}(x - yh) - \rho_{\sharp}(x) = \sum_{\ell=1}^{p-1} \frac{\rho_{\sharp}^{(\ell)}(x)}{\ell!} (-yh)^\ell + \frac{\rho_{\sharp}^{(p)}(x - \theta yh)}{p!} (-yh)^p$$

for some $\theta \in [0, 1]$, where $\sum_{\ell=1}^0 = 0$ by convention. Since $\rho_{\sharp}^{(p)}$ is $(r-p)$ -Hölder continuous, we have that $H := \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|\rho_{\sharp}^{(p)}(x) - \rho_{\sharp}^{(p)}(y)|}{|x-y|^{r-p}} < \infty$. Now, since $\int_{\mathbb{R}} y^\ell W(y) dy = 0$ for $\ell = 1, \dots, p$, we have that for any $x \in \mathbb{R}$,

$$\begin{aligned} \left| \int_{\mathbb{R}} \{\rho_{\sharp}(x - yh) - \rho_{\sharp}(x)\} W(y) dy \right| &= \left| \int_{\mathbb{R}} \left[\{\rho_{\sharp}(x - yh) - \rho_{\sharp}(x)\} - \sum_{\ell=1}^p \frac{\rho_{\sharp}^{(\ell)}(x)}{\ell!} (-yh)^\ell \right] W(y) dy \right| \\ &\leq \frac{Hh^r}{p!} \int_{\mathbb{R}} |y|^r |W(y)| dy, \end{aligned}$$

where $0! = 1$ by convention. This completes the proof. \square

7.2. **Proof of Theorem 4.1.** Observe that

$$\begin{aligned}
x^2(\widehat{\rho}(x) - \rho(x)) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (-\psi''(u) - \sigma^2) \varphi_W(uh) du - x^2 \rho(x) \\
&\quad + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (\psi(u) - \widehat{\psi}(u))'' \varphi_W(uh) du + (\sigma^2 - \widehat{\sigma}^2) \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \varphi_W(uh) du \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \underbrace{\left(\int_{\mathbb{R}} e^{iuy} y^2 \rho(y) dy \right)}_{=[(y^2 \rho) * (h^{-1} W(\cdot/h))](x)} \varphi_W(uh) du - x^2 \rho(x) \\
&\quad + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (\psi(u) - \widehat{\psi}(u))'' \varphi_W(uh) du \\
&\quad + (\sigma^2 - \widehat{\sigma}^2) h^{-1} W(x/h) \\
&=: II_{1,n} + II_{2,n} + III_{3,n}. \tag{7.5}
\end{aligned}$$

For the first and third terms, we have that $\|II_{1,n}\|_I \lesssim h^r \ll (n\Delta h \log n)^{-1/2}$ (by Lemma 7.9) and $\|III_{3,n}\|_I \ll (n\Delta h \log n)^{-1/2}$ under our assumption. For the second term $II_{2,n}$, Lemma 7.5 yields that

$$II_{2,n} = \frac{-1}{2\pi\Delta} \int_{\mathbb{R}} e^{-iux} \frac{(\widehat{\varphi}_\Delta - \varphi_\Delta)''}{\varphi_\Delta}(u) \varphi_W(uh) du + o_P\{(n\Delta h \log n)^{-1/2}\}$$

uniformly in $x \in I$, and observe that the first term on the right hand side can be expressed as

$$\frac{1}{n\Delta h} \sum_{j=1}^n \{Y_{n,j}^2 K_n((x - Y_{n,j})/h) - \mathbb{E}[Y_{n,1}^2 K_n((x - Y_{n,1})/h)]\}.$$

Therefore, since $\inf_{x \in I} s_n(x) \gtrsim \sqrt{\Delta h}$, we have that

$$\frac{\sqrt{n}\Delta h x^2(\widehat{\rho}(x) - \rho(x))}{s_n(x)} = Z_n(x) + o_P\{(\log n)^{-1/2}\} \tag{7.6}$$

uniformly in $x \in I$.

Now, we approximate $\|Z_n\|_I$ by the supremum in absolute value of a tight Gaussian random variable in $\ell^\infty(I)$ with mean zero and the same covariance function as Z_n . To this end, we shall employ Theorem 2.1 in [18]. Consider the empirical process

$$\mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \{f(Y_{n,j}) - \mathbb{E}[f(Y_{n,1})]\}, \quad f \in \mathcal{F}_n.$$

The covering number bound (7.4) ensures the existence of a tight Gaussian random variable \mathbb{U}_n in $\ell^\infty(\mathcal{F}_n)$ with mean zero and the same covariance function as $\{\mathbb{G}_n(f) : f \in \mathcal{F}_n\}$. Extend \mathbb{G}_n linearly to $\mathcal{F}_n \cup (-\mathcal{F}_n) = \{f, -f : f \in \mathcal{F}_n\}$, and observe that $\|\mathbb{G}_n\|_{\mathcal{F}_n \cup (-\mathcal{F}_n)} = \sup_{f \in \mathcal{F}_n \cup (-\mathcal{F}_n)} \mathbb{G}_n(f)$. Note that from Theorem 3.7.28 in [34], \mathbb{U}_n extends to the linear hull of \mathcal{F}_n in such a way that \mathbb{U}_n has linear sample paths, so that $\|\mathbb{U}_n\|_{\mathcal{F}_n \cup (-\mathcal{F}_n)} = \sup_{f \in \mathcal{F}_n \cup (-\mathcal{F}_n)} \mathbb{U}_n(f)$, and in addition \mathbb{U}_n has uniformly continuous paths on the symmetric convex hull of \mathcal{F}_n . It is not difficult to verify that the covering number of $\mathcal{F}_n \cup (-\mathcal{F}_n)$ is at most twice that of \mathcal{F}_n . In particular, $\{\mathbb{U}_n(f) : f \in \mathcal{F}_n \cup (-\mathcal{F}_n)\}$ is a tight Gaussian random variable in $\ell^\infty(\mathcal{F}_n \cup (-\mathcal{F}_n))$ with mean zero and the same covariance function as $\{\mathbb{G}_n(f) : f \in \mathcal{F}_n \cup (-\mathcal{F}_n)\}$.

Next, since $Y_{n,1} - b\Delta$ has distribution $P_{\Delta,b}$ such that $y^4 P_{\Delta,b}(dy) = g_{\Delta,b}(y)dy$ and $\|g_{\Delta,b}\|_{\mathbb{R}} \lesssim \Delta$, we have that

$$\begin{aligned} \mathbb{E}[Y_{n,1}^4 K_n^2((x - Y_{n,1})/h)] &\leq 8\mathbb{E}[(Y_{n,1} - b\Delta)^4 K_n^2((x - (Y_{n,1} - b\Delta) - b\Delta)/h)] + 8b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2 \\ &= 8 \int_{\mathbb{R}} K_n^2((x - b\Delta - y)/h) g_{\Delta,b}(y) dy + 8b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2 \\ &= 8h \int_{\mathbb{R}} K_n^2(y) g_{\Delta,b}(x - b\Delta - yh) dy + 8b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2 \\ &\leq 8h \|g_{\Delta,b}\|_{\mathbb{R}} \int_{\mathbb{R}} K_n^2(y) dy + 8b^4 \Delta^4 \|K_n\|_{\mathbb{R}}^2 \lesssim \Delta h \int_{\mathbb{R}} \left| \frac{\varphi_W(u)}{\varphi_{\Delta}(u/h)} \right|^2 du + \Delta^4 \lesssim \Delta h, \end{aligned} \quad (7.7)$$

so that $\sup_{f \in \mathcal{F}_n} \mathbb{E}[f^2(Y_{n,1})] \lesssim 1$. On the other hand, since $\sup_{f \in \mathcal{F}_n} \|f\|_{\mathbb{R}} \lesssim 1/\sqrt{\Delta h}$,

$$\begin{aligned} \sup_{f \in \mathcal{F}_n} \mathbb{E}[|f(Y_{n,1})|^3] &\lesssim \sup_{f \in \mathcal{F}_n} \mathbb{E}[f^2(Y_{n,1})]/\sqrt{\Delta h} \lesssim 1/\sqrt{\Delta h} \quad \text{and} \\ \sup_{f \in \mathcal{F}_n} \mathbb{E}[f^4(Y_{n,1})] &\lesssim \sup_{f \in \mathcal{F}_n} \mathbb{E}[f^2(Y_{n,1})]/(\Delta h) \lesssim 1/(\Delta h). \end{aligned}$$

Therefore, applying Theorem 2.1 in [18] to $\mathcal{F}_n \cup (-\mathcal{F}_n)$ with $B(f) \equiv 0$, $A \lesssim 1$, $v \lesssim 1$, $\sigma \sim 1$, $b \lesssim 1/\sqrt{\Delta h}$, $\gamma \lesssim 1/\log n$, and sufficiently large q (in the notation used in the cited theorem), yields that there exists a random variable V_n with $V_n \stackrel{d}{=} \|\mathbb{U}_n\|_{\mathcal{F}_n}$ such that

$$\left| \|\mathbb{G}_n\|_{\mathcal{F}_n} - V_n \right| = O_{\mathbb{P}} \left\{ \frac{(\log n)^{1+1/q}}{n^{1/2-1/q}\sqrt{\Delta h}} + \frac{\log n}{(n\Delta h)^{1/6}} \right\} = o_{\mathbb{P}}\{(\log n)^{-1/2}\}. \quad (7.8)$$

Now, for $f_{n,x}(y) = y^2 K_n((x - y)/h)/s_n(x)$, define

$$Z_n^G(x) = \mathbb{U}_n(f_{n,x}), \quad x \in I,$$

and observe that Z_n^G is a tight Gaussian random variable in $\ell^\infty(I)$ with mean zero and the same covariance function as Z_n . We will derive the conclusion of the theorem from (7.8). To this end, the following anti-concentration inequality will play a crucial role: for any $\varepsilon > 0$,

$$\sup_{z \in \mathbb{R}} \mathbb{P}\{\|Z_n^G\|_I - z \leq \varepsilon\} \leq 4\varepsilon(1 + \mathbb{E}[\|Z_n^G\|_I]). \quad (7.9)$$

See Corollary 2.1 in [16]; see also Theorem 3 in [17]. From the result (7.8), there exists a sequence of constants $\varepsilon_n \downarrow 0$ such that $\mathbb{P}\{\|\mathbb{G}_n\|_{\mathcal{F}_n} - V_n > \varepsilon_n(\log n)^{-1/2}\} \leq \varepsilon_n$. Since $\|\mathbb{G}_n\|_{\mathcal{F}_n} = \|Z_n\|_I$ and $V_n \stackrel{d}{=} \|Z_n^G\|_I$, we have that

$$\begin{aligned} \mathbb{P}\{\|Z_n\|_I \leq z\} &\leq \mathbb{P}\{\|Z_n^G\|_I \leq z + \varepsilon_n(\log n)^{-1/2}\} + \varepsilon_n \\ &\leq \mathbb{P}\{\|Z_n^G\|_I \leq z\} + 4\varepsilon_n(\log n)^{-1/2}(1 + \mathbb{E}[\|Z_n^G\|_I]) + \varepsilon_n \end{aligned}$$

for any $z \in \mathbb{R}$. Likewise, we have

$$\mathbb{P}\{\|Z_n\|_I \leq z\} \geq \mathbb{P}\{\|Z_n^G\|_I \leq z\} - 4\varepsilon_n(\log n)^{-1/2}(1 + \mathbb{E}[\|Z_n^G\|_I]) - \varepsilon_n$$

for any $z \in \mathbb{R}$. Now, from the covering number bound (7.4) together with the fact that $\text{Var}(f_{n,x}(Y_{n,1})) = 1$ for all $x \in I$, Dudley's entropy integral bound [cf. 63, Corollary 2.2.8] yields that

$$\mathbb{E}[\|Z_n^G\|_I] = \mathbb{E}[\|\mathbb{U}_n\|_{\mathcal{F}_n}] \lesssim \int_0^1 \sqrt{1 + \log(1/(\varepsilon\sqrt{\Delta h}))} d\varepsilon \lesssim \sqrt{\log n}.$$

This completes the proof.

□

7.3. Proof of Theorem 4.2. We first prove the following technical lemma.

Lemma 7.10. $\|\widehat{s}_n^2/s_n^2 - 1\|_I = o_P\{(\log n)^{-1}\}$.

Proof. From Lemma 7.4, we have that $\|\widehat{\varphi}_\Delta - \varphi_\Delta\|_{[-h^{-1}, h^{-1}]} = O_P(n^{-1/2} \log n)$, so that it is not difficult to verify that $\|\widehat{K}_n - K_n\|_{\mathbb{R}} = O_P(n^{-1/2} \log n)$ and $\|\widehat{K}_n^2 - K_n^2\|_{\mathbb{R}} \leq \|\widehat{K}_n + K_n\|_{\mathbb{R}} \|\widehat{K}_n - K_n\|_{\mathbb{R}} = O_P(n^{-1/2} \log n)$. Hence

$$\begin{aligned} & \left\| \frac{1}{n} \sum_{j=1}^n Y_{n,j}^2 \{ \widehat{K}_n((\cdot - Y_{n,j})/h) - K_n((\cdot - Y_{n,j})/h) \} \right\|_I \\ & \leq \underbrace{\left(\frac{1}{n} \sum_{j=1}^n Y_{n,j}^2 \right)}_{=O_P(\Delta)} \|\widehat{K}_n - K_n\|_{\mathbb{R}} = O_P(n^{-1/2} \Delta \log n), \end{aligned} \quad (7.10)$$

and likewise

$$\left\| \frac{1}{n} \sum_{j=1}^n Y_{n,j}^4 \{ \widehat{K}_n^2((\cdot - Y_{n,j})/h) - K_n^2((\cdot - Y_{n,j})/h) \} \right\|_I = O_P(n^{-1/2} \Delta \log n).$$

Since $\|n^{-1} \sum_{j=1}^n Y_{n,j}^2 K_n((x - Y_{n,j})/h)\|_I \lesssim n^{-1} \sum_{j=1}^n Y_{n,j}^2 = O_P(\Delta)$, we have that

$$\widehat{s}_n^2(x) = \underbrace{\frac{1}{n} \sum_{j=1}^n Y_{n,j}^4 K_n^2((x - Y_{n,j})/h) - \left\{ \frac{1}{n} \sum_{j=1}^n Y_{n,j}^2 K_n((x - Y_{n,j})/h) \right\}^2}_{=: \widetilde{s}_n^2(x)} + O_P(n^{-1/2} \Delta \log n)$$

uniformly in $x \in I$. Furthermore, since $\inf_{x \in I} s_n^2(x) \gtrsim \Delta h$ and

$$\frac{n^{-1/2} \Delta \log n}{\Delta h} = n^{-1/2} h^{-1} \log n \lesssim n^{-1/2} \Delta^{-1/2} \log n \ll (\log n)^{-1},$$

it remains to prove that $\|\widetilde{s}_n^2/s_n^2 - 1\|_I = o_P\{(\log n)^{-1}\}$. To this end, since $\|E[Y_{n,1}^2 K_n((\cdot - Y_{n,1})/h)]/s_n\|_I \lesssim \Delta/\sqrt{\Delta h} = \sqrt{\Delta/h} \lesssim \Delta^{1/4}$, it suffices to prove that

$$\left\| \frac{1}{n} \sum_{j=1}^n \{f^2(Y_{n,j}) - E[f^2(Y_{n,1})]\} \right\|_{\mathcal{F}_n} = o_P\{(\log n)^{-1}\}, \text{ and} \quad (7.11)$$

$$\left\| \frac{1}{n} \sum_{j=1}^n \{f(Y_{n,j}) - E[f(Y_{n,1})]\} \right\|_{\mathcal{F}_n} = o_P\{(\log n)^{-1/2}\}. \quad (7.12)$$

To prove (7.11), we make use of Corollary 5.1 in [15]. Let $\mathcal{F}_n^2 = \{f^2 : f \in \mathcal{F}_n\}$. Observe that $\sup_{f \in \mathcal{F}_n^2} E[f^2(Y_{n,1})] = \sup_{f \in \mathcal{F}_n} E[f_n^4(Y_{n,1})] \lesssim 1/(\Delta h)$, and $\sup_{f \in \mathcal{F}_n^2} \|f\|_{\mathbb{R}} = \sup_{f \in \mathcal{F}_n} \|f^2\|_{\mathbb{R}} \lesssim 1/(\Delta h)$. From the covering number bound (7.4) together with Corollary A.1 in [15], there exist constants $A_2, v_2 > 0$ independent of n such that

$$\sup_Q N(\mathcal{F}_n^2, \|\cdot\|_{Q,2}, \varepsilon/(\Delta h)) \leq (A_2/\varepsilon)^{v_2}, \quad 0 < \forall \varepsilon \leq 1,$$

where \sup_Q is taken over all finitely discrete distributions on \mathbb{R} . Therefore, from Corollary 5.1 in [15], the expectation of the left hand side on (7.11) is bounded by

$$\lesssim \sqrt{\frac{\log n}{n\Delta h}} + \frac{\log n}{n\Delta h} \ll (\log n)^{-1}.$$

For (7.12), from the covering number bound (7.4), Theorem 2.14.1 in [63] shows that the expectation of the left hand side on (7.12) is bounded by

$$\lesssim (n\Delta h)^{-1/2} \ll (\log n)^{-1/2}.$$

This completes the proof. \square

Proof of Theorem 4.2. We separately prove the theorem for the multiplier and empirical bootstraps.

Multiplier bootstrap case: We first verify that

$$\begin{aligned} & \sum_{j=1}^n \xi_j \left\{ Y_{n,j}^2 \widehat{K}_n((x - Y_{n,j})/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 \widehat{K}_n((x - Y_{n,j'})/h) \right\} \\ &= \sum_{j=1}^n \xi_j \left\{ Y_{n,j}^2 K_n((x - Y_{n,j})/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 K_n((x - Y_{n,j'})/h) \right\} \\ & \quad + o_{\mathbb{P}}(\sqrt{n\Delta h / \log n}) \end{aligned} \tag{7.13}$$

uniformly in $x \in I$. From (7.10),

$$\left\| \left(\sum_{j=1}^n \xi_j \right) \left[\frac{1}{n} \sum_{j=1}^n Y_{n,j}^2 \{ \widehat{K}_n((\cdot - Y_{n,j})/h) - K_n((\cdot - Y_{n,j})/h) \} \right] \right\|_I = O_{\mathbb{P}}(\Delta \log n).$$

Next, observe that

$$\begin{aligned} & \left\| \sum_{j=1}^n \xi_j Y_{n,j}^2 \{ \widehat{K}_n((\cdot - Y_{n,j})/h) - K_n((\cdot - Y_{n,j})/h) \} \right\|_I \\ & \lesssim \int_{-1}^1 \left| \sum_{j=1}^n \xi_j Y_{n,j}^2 e^{iuY_{n,j}/h} \right| \left| \frac{1}{\widehat{\varphi}_{\Delta}(u/h)} - \frac{1}{\varphi_{\Delta}(u/h)} \right| du \\ & \leq O_{\mathbb{P}}(n^{-1/2} \log n) \int_{-1}^1 \left| \sum_{j=1}^n \xi_j Y_{n,j}^2 e^{iuY_{n,j}/h} \right| du, \text{ and} \\ & \mathbb{E} \left[\left| \sum_{j=1}^n \xi_j Y_{n,j}^2 e^{iuY_{n,j}/h} \right| \right] \leq \mathbb{E} \left[\sqrt{\sum_{j=1}^n Y_{n,j}^4} \right] \lesssim \sqrt{n\Delta}, \end{aligned}$$

which yields that

$$\left\| \sum_{j=1}^n \xi_j Y_{n,j}^2 \{ \widehat{K}_n((\cdot - Y_{n,j})/h) - K_n((\cdot - Y_{n,j})/h) \} \right\|_I = O_{\mathbb{P}}(\sqrt{\Delta} \log n).$$

Therefore, we have proved (7.13).

Now, since $\|1/\widehat{s}_n\|_I = O_{\mathbb{P}}(1/\sqrt{\Delta h})$ by Lemma 7.10, we have that

$$\begin{aligned} \widehat{Z}_n^{MB}(x) &= \frac{1}{\widehat{s}_n(x)\sqrt{n}} \sum_{j=1}^n \xi_j \left\{ Y_{n,j}^2 K_n((x - Y_{n,j})/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 K_n((x - Y_{n,j'})/h) \right\} \\ &\quad + o_{\mathbb{P}}\{(\log n)^{-1/2}\} \\ &= [1 + o_{\mathbb{P}}\{(\log n)^{-1}\}] \underbrace{\frac{1}{s_n(x)\sqrt{n}} \sum_{j=1}^n \xi_j \left\{ Y_{n,j}^2 K_n((x - Y_{n,j})/h) - n^{-1} \sum_{j'=1}^n Y_{n,j'}^2 K_n((x - Y_{n,j'})/h) \right\}}_{=: Z_n^{MB}(x)} \\ &\quad + o_{\mathbb{P}}\{(\log n)^{-1/2}\} \end{aligned} \tag{7.14}$$

uniformly in $x \in I$. We wish to show that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|Z_n^{MB}\|_I \leq z \mid \mathcal{D}_n\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \xrightarrow{\mathbb{P}} 0. \tag{7.15}$$

To this end, we shall apply Theorem 2.2 in [18] to $\mathcal{F}_n \cup (-\mathcal{F}_n)$. Let

$$\mathbb{G}_n^\xi(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \xi_j \left\{ f(Y_{n,j}) - n^{-1} \sum_{j'=1}^n f(Y_{n,j'}) \right\}, \quad f \in \mathcal{F}_n.$$

Applying Theorem 2.2 in [18] to $\mathcal{F}_n \cup (-\mathcal{F}_n)$ with $B(f) \equiv 0$, $A \lesssim 1$, $v \lesssim 1$, $\sigma \sim 1$, $b \lesssim 1/\sqrt{\Delta h}$, $\gamma \lesssim 1/\log n$, and sufficiently large q (in the notation used in the cited theorem), yields that there exists a random variable V_n^ξ whose conditional distribution given \mathcal{D}_n is identical to the distribution of $\|\mathbb{U}_n\|_{\mathcal{F}_n} (= \|Z_n^G\|_I)$, i.e., $\mathbb{P}\{V_n^\xi \leq z \mid \mathcal{D}_n\} = \mathbb{P}\{\|Z_n^G\|_I \leq z\}$ for all $z \in \mathbb{R}$ almost surely, and such that

$$\left\| \|\mathbb{G}_n^\xi\|_{\mathcal{F}_n} - V_n^\xi \right\| = O_{\mathbb{P}} \left\{ \frac{(\log n)^{2+1/q}}{n^{1/2-1/q}\sqrt{\Delta h}} + \frac{(\log n)^{7/4+1/q}}{(n\Delta h)^{1/4}} \right\} = o_{\mathbb{P}}\{(\log n)^{-1/2}\},$$

which implies that there exists a sequence of constants $\varepsilon_n \downarrow 0$ such that

$$\mathbb{P}\{\|\mathbb{G}_n^\xi\|_{\mathcal{F}_n} - V_n^\xi > \varepsilon_n(\log n)^{-1/2} \mid \mathcal{D}_n\} \xrightarrow{\mathbb{P}} 0.$$

Since $\|\mathbb{G}_n^\xi\|_{\mathcal{F}_n} = \|Z_n^{MB}\|_I$, we have that

$$\begin{aligned} \mathbb{P}\{\|Z_n^{MB}\|_I \leq z \mid \mathcal{D}_n\} &\leq \mathbb{P}\{V_n^\xi \leq z + \varepsilon_n(\log n)^{-1/2} \mid \mathcal{D}_n\} + o_{\mathbb{P}}(1) \\ &= \mathbb{P}\{\|Z_n^G\|_I \leq z + \varepsilon_n(\log n)^{-1/2}\} + o_{\mathbb{P}}(1) \end{aligned}$$

uniformly in $z \in \mathbb{R}$. From the anti-concentration inequality (7.9) together with the bound $\mathbb{E}[\|Z_n^G\|_I] \lesssim \sqrt{\log n}$, we conclude that

$$\mathbb{P}\{\|Z_n^{MB}\|_I \leq z \mid \mathcal{D}_n\} \leq \mathbb{P}\{\|Z_n^G\|_I \leq z\} + o_{\mathbb{P}}(1)$$

uniformly in $z \in \mathbb{R}$. Likewise, we have $\mathbb{P}\{\|Z_n^{MB}\|_I \leq z \mid \mathcal{D}_n\} \geq \mathbb{P}\{\|Z_n^G\|_I \leq z\} - o_{\mathbb{P}}(1)$ uniformly in $z \in \mathbb{R}$. Hence, we have proved (7.15).

From (7.15) together with the bound $\mathbb{E}[\|Z_n^G\|_I] \lesssim \sqrt{\log n}$, we see that $\|Z_n^{MB}\|_I = O_{\mathbb{P}}(\sqrt{\log n})$. So, from (7.14), we have that

$$\widehat{Z}_n^{MB}(x) = Z_n^{MB}(x) + o_{\mathbb{P}}\{(\log n)^{-1/2}\}$$

uniformly in $x \in I$. In view of the proof of (7.15), we conclude that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|\widehat{Z}_n^{MB}\|_I \leq z \mid \mathcal{D}_n\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \xrightarrow{\mathbb{P}} 0. \quad (7.16)$$

Now, we wish to show that $\mathbb{P}\{\rho(x) \in \widehat{\mathcal{C}}_{1-\tau}^{MB}(x) \forall x \in I\} \rightarrow 1 - \tau$. We begin with noting that

$$\rho(x) \in \widehat{\mathcal{C}}_{1-\tau}^{MB} \forall x \in I \Leftrightarrow \|\sqrt{n}\Delta h x^2(\widehat{\rho} - \rho)/\widehat{s}_n\|_I \leq \widehat{c}_n^{MB}(1 - \tau).$$

Recall that from the conclusion of Theorem 4.1 together with the bound $\mathbb{E}[\|Z_n^G\|_I] \lesssim \sqrt{\log n}$, we have that $\|Z_n\|_I = O_{\mathbb{P}}(\sqrt{\log n})$. Observe that

$$\begin{aligned} \frac{\sqrt{n}\Delta h x^2(\widehat{\rho}(x) - \rho(x))}{\widehat{s}_n(x)} &= \frac{s_n(x)}{\widehat{s}_n(x)} \cdot \frac{\sqrt{n}\Delta h x^2(\widehat{\rho}(x) - \rho(x))}{s_n(x)} \\ &= \frac{s_n(x)}{\widehat{s}_n(x)} \left[Z_n(x) + o_{\mathbb{P}}\{(\log n)^{-1/2}\} \right] \quad (\text{from (7.6)}) \\ &= [1 + o_{\mathbb{P}}\{(\log n)^{-1}\}] \left[Z_n(x) + o_{\mathbb{P}}\{(\log n)^{-1/2}\} \right] \quad (\text{from Lemma 7.10}) \\ &= Z_n(x) + o_{\mathbb{P}}\{(\log n)^{-1/2}\} \quad (\text{from } \|Z_n\|_I = O_{\mathbb{P}}(\sqrt{\log n})) \end{aligned}$$

uniformly in $x \in I$. Hence, using the conclusion of Theorem 4.1 together with the anti-concentration inequality (7.9), we have that

$$\sup_{z \in \mathbb{R}} |\mathbb{P}\{\|\sqrt{n}\Delta h x^2(\widehat{\rho} - \rho)/\widehat{s}_n\|_I \leq z\} - \mathbb{P}\{\|Z_n^G\|_I \leq z\}| \rightarrow 0. \quad (7.17)$$

From the result (7.16), using an argument similar to Step 3 in the proof of Theorem 2 in [45], we can find a sequence of constants $\varepsilon'_n \downarrow 0$ such that

$$c_n^G(1 - \tau - \varepsilon'_n) \leq \widehat{c}_n^{MB}(1 - \tau) \leq c_n^G(1 - \tau + \varepsilon'_n) \quad (7.18)$$

with probability approaching one. Therefore,

$$\begin{aligned} &\mathbb{P}\{\|\sqrt{n}\Delta h x^2(\widehat{\rho} - \rho)/\widehat{s}_n\|_I \leq \widehat{c}_n^{MB}(1 - \tau)\} \\ &\leq \mathbb{P}\{\|\sqrt{n}\Delta h x^2(\widehat{\rho} - \rho)/\widehat{s}_n\|_I \leq c_n^G(1 - \tau + \varepsilon'_n)\} + o(1) \quad (\text{from (7.18)}) \\ &= \underbrace{\mathbb{P}\{\|Z_n^G\|_I \leq c_n^G(1 - \tau + \varepsilon'_n)\}}_{=1-\tau+\varepsilon'_n} + o(1) \quad (\text{from (7.17)}) \\ &= 1 - \tau + o(1). \end{aligned}$$

Note that from the anti-concentration inequality, the distribution function of $\|Z_n^G\|_I$ is continuous, so that the equality $\mathbb{P}\{\|Z_n^G\|_I \leq c_n^G(1 - \tau + \varepsilon'_n)\} = 1 - \tau + \varepsilon'_n$ holds. Likewise, we have $\mathbb{P}\{\|\sqrt{n}\Delta h x^2(\widehat{\rho} - \rho)/\widehat{s}_n\|_I \leq \widehat{c}_n^{MB}(1 - \tau)\} \geq 1 - \tau - o(1)$.

Finally, the Borell-Sudakov-Tsirelson inequality [63, Lemma A.2.2] yields that

$$c_n^G(1 - \tau + \varepsilon'_n) \lesssim \mathbb{E}[\|Z_n^G\|_I] + \sqrt{\log(1/(\tau - \varepsilon'_n))} \lesssim \sqrt{\log n},$$

which implies that $\widehat{c}_n^{MB}(1 - \tau) = O_{\mathbb{P}}(\sqrt{\log n})$ from (7.18). Therefore, the supremum width of the band $\widehat{\mathcal{C}}_{1-\tau}^{MB}$ is

$$2 \sup_{x \in I} \frac{\widehat{s}_n(x)}{x^2 \sqrt{n}\Delta h} \widehat{c}_n^{MB}(1 - \tau) \lesssim \{1 + o_{\mathbb{P}}(1)\} \frac{\sup_{x \in I} s_n(x)}{\sqrt{n}\Delta h} \widehat{c}_n^{MB}(1 - \tau) = O_{\mathbb{P}}\{(n\Delta h)^{-1/2} \sqrt{\log n}\},$$

where the bound $\sup_{x \in I} s_n(x) \lesssim \sqrt{\Delta h}$ follows from (7.7). This completes the proof for the multiplier bootstrap case.

Empirical bootstrap case: Note that the bootstrap process \widehat{Z}_n^{EB} can be expressed as

$$\widehat{Z}_n^{EB}(x) = \frac{1}{\widehat{s}_n(x)\sqrt{n}} \sum_{j=1}^n (M_{n,j} - 1) Y_{n,j}^2 \widehat{K}_n((x - Y_{n,j})/h), \quad x \in I$$

where each $M_{n,j}$ is the number of times that $Y_{n,j}$ is “redrawn” in the bootstrap sample, and the vector $(M_{n,1}, \dots, M_{n,n})$ is multinomially distributed with parameters n and (probabilities) $(1/n, \dots, 1/n)$ independent of the data \mathcal{D}_n [cf. 63, Section 3.6]. Given this expression, the proof for the empirical bootstrap case is almost identical to that for the multiplier bootstrap case, where we use Theorem 2.3 in [18] instead of their Theorem 2.2. We omit the detail for brevity. \square

8. ADDITIONAL PROOFS

8.1. Proof of Lemma 2.1. The Lévy process L has the following decomposition (Lévy-Itô decomposition)

$$L_t = \gamma t + \sigma B_t + \int_0^t \int_{\mathbb{R}} x 1_{[-1,1]}(x) (\mu^L - \nu^L)(ds, dx) + \int_0^t \int_{\mathbb{R}} x 1_{[-1,1]^c}(x) \mu^L(ds, dx),$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion and μ^L is a Poisson random measure on $[0, \infty) \times \mathbb{R}$, independent of B , with intensity measure $\nu^L(dt, dx) = dt\nu(dx)$. Depending on the value of $\alpha \in (0, 2)$, we set

$$L^{(1)} = L - L^{(2)}, \quad L_t^{(2)} = \begin{cases} \int_0^t \int_{\mathbb{R}} x \mu^L(ds, dx) & \text{if } \alpha \in (0, 1) \\ \int_0^t \int_{\mathbb{R}} x (\mu^L - \nu^L)(ds, dx) & \text{if } \alpha \in [1, 2) \end{cases}.$$

Let $L_{n,j}^{(1)} = L_{j\Delta}^{(1)} - L_{(j-1)\Delta}^{(1)}$. Consider the following decomposition for $\widehat{\sigma}_{TRV}^2 - \sigma^2$:

$$\begin{aligned} \widehat{\sigma}_{TRV}^2 - \sigma^2 &= \underbrace{\frac{1}{n\Delta} \sum_{j=1}^n (L_{n,j}^{(1)})^2 - \sigma^2}_{=: A_{1,n}} - \underbrace{\frac{1}{n\Delta} \sum_{j=1}^n (L_{n,j}^{(1)})^2 1_{\{|L_{n,j}^{(1)}| > \alpha_0 \Delta^{\theta_0}\}}}_{=: A_{2,n}} \\ &\quad + \underbrace{\widehat{\sigma}_{TRV}^2 - \frac{1}{n\Delta} \sum_{j=1}^n (L_{n,j}^{(1)})^2 1_{\{|L_{n,j}^{(1)}| \leq \alpha_0 \Delta^{\theta_0}\}}}_{=: A_{3,n}}. \end{aligned}$$

We shall evaluate $A_{k,n}$ for $k = 1, 2, 3$. Since $L_{n,j}^{(1)} \sim N(\Delta\gamma_0, \Delta\sigma^2)$ with

$$\gamma_0 = \begin{cases} \gamma - \int_{[-1,1]} x \nu(dx) & \text{if } \alpha \in (0, 1) \\ \gamma + \int_{[-1,1]^c} x \nu(dx) & \text{if } \alpha \in [1, 2) \end{cases},$$

we have that

$$\mathbb{E}[(L_{n,j}^{(1)})^2] = \Delta^2 \gamma_0^2 + \Delta \sigma^2, \quad \mathbb{E}[(L_{n,j}^{(1)})^4] = \Delta^4 \gamma_0^4 + 6\Delta^3 \gamma_0^2 \sigma^2 + 3\Delta^2 \sigma^4,$$

which yields that

$$A_{1,n} = \underbrace{\frac{1}{n\Delta} \sum_{j=1}^n \left\{ (L_{n,j}^{(1)})^2 - \mathbb{E}[(L_{n,j}^{(1)})^2] \right\}}_{=O_{\mathbb{P}}(n^{-1/2})} + \underbrace{\frac{1}{n\Delta} \sum_{j=1}^n \mathbb{E}[(L_{n,j}^{(1)})^2] - \sigma^2}_{=O(\Delta)} = O_{\mathbb{P}}(n^{-1/2} + \Delta).$$

Furthermore, for any $q > 0$, we have that $\mathbb{E}[|L_{n,j}^{(1)}|^q] \lesssim \Delta^{q/2}$, which yields that

$$\mathbb{E} \left[(L_{n,j}^{(1)})^2 \mathbf{1}_{\{|L_{n,j}^{(1)}| > \alpha_0 \Delta^{\theta_0}\}} \right] \lesssim \Delta^{1+q(1-2\theta_0)/2}.$$

Taking $q = 2/(1 - 2\theta_0)$, we have that $\mathbb{E}[|A_{2,n}|] \lesssim \Delta$. Finally, applying Lemma 13.2.6 in [39] with $k = 1, F(x) = x^2, s' = 2, m = s = p' = 1$, and $\theta = 0$ in the notation used in the lemma, we conclude that $\mathbb{E}[|A_{3,n}|] \lesssim \Delta^{(2-\alpha)\theta_0}$. These estimates yield the desired result. \square

8.2. Proof of Proposition 4.1. Since $L_t - bt$ is also Lévy process with the same Lévy density ρ as L_t , without loss of generality, we may assume $b = 0$, and we shall verify Condition (ii) for $P_{\Delta,b} = P_{\Delta}$. In this proof, slightly abusing notation, we shall use the same symbol for a measure and its Lebesgue density if the latter exists. Let $\|\cdot\|_{L^1}$ denote the L^1 -norm with respect to the Lebesgue measure. We first note that since yP_{Δ} is absolutely continuous, for any integer $\ell > 1$, $y^{\ell}P_{\Delta}$ is also absolutely continuous with density $(y^{\ell}P_{\Delta})(y) = y^{\ell-1}((yP_{\Delta})(y))$ (this means that the density of the signed measure $y^{\ell}P_{\Delta}(dy)$ is given by the multiple of $y^{\ell-1}$ with the Lebesgue density of the signed measure $yP_{\Delta}(dy)$). Furthermore, boundedness of $x^4\rho$ ensures that $\int_{[-1,1]^c} x^2\rho(x)dx < \infty$, which in turn ensures that P_{Δ} has finite second moment. In what follows, we will freely use some basic results on convolutions and Fourier transforms of finite signed Borel measures on \mathbb{R} ; cf. Folland [33, Section 8.6].

Verification of $\|y^4P_{\Delta}\|_{\mathbb{R}} \lesssim \Delta$: The proof is similar to that of Proposition 14 in [56]. We first note that $\varphi_{\Delta} = \varphi_{\Delta/2}^2$ by infinite divisibility. Using this property, we have that $\varphi_{\Delta}'' = \{\Delta\psi'' + (\Delta\psi'')^2\}\varphi_{\Delta} = \Delta\psi''\varphi_{\Delta} + 4\{(\Delta/2)\psi'\varphi_{\Delta/2}\}^2 = \Delta\psi''\varphi_{\Delta} + 4\{(\varphi_{\Delta/2})'\}^2$, i.e.,

$$\int_{\mathbb{R}} e^{iuy} y^2 P_{\Delta}(dy) = \Delta \underbrace{\left(\sigma^2 + \int_{\mathbb{R}} e^{iuy} y^2 \nu(dy) \right)}_{= \int_{\mathbb{R}} e^{ity} (\sigma^2 \delta_0 + y^2 \nu)(dy)} \int_{\mathbb{R}} e^{iuy} P_{\Delta}(dy) + 4 \left(\int_{\mathbb{R}} e^{iuy} y P_{\Delta/2}(dy) \right)^2.$$

Applying the Fourier inversion, we have that

$$y^2 P_{\Delta} = \Delta \nu_{\sigma} * P_{\Delta} + 4(yP_{\Delta/2}) * (yP_{\Delta/2}), \quad (8.1)$$

where $\nu_{\sigma} = \sigma^2 \delta_0 + y^2 \nu$ and $*$ denotes the convolution. Using the rule $y(P*Q) = (yP)*Q + P*(yQ)$ for finite signed Borel measures P, Q on \mathbb{R} such that $\int_{\mathbb{R}} |y||P|(dy) < \infty$ and $\int_{\mathbb{R}} |y||Q|(dy) < \infty$, we have that

$$\begin{aligned} y^3 P_{\Delta} &= \Delta \{ (y^3 \nu) * P_{\Delta} + \nu_{\sigma} * (yP_{\Delta}) \} + 8(yP_{\Delta/2}) * (y^2 P_{\Delta/2}), \text{ and} \\ y^4 P_{\Delta} &= \Delta \{ (y^4 \nu) * P_{\Delta} + 2(y^3 \nu) * (yP_{\Delta}) + \nu_{\sigma} * (y^2 P_{\Delta}) \} \\ &\quad + 8(y^3 P_{\Delta/2}) * (yP_{\Delta/2}) + 8(y^2 P_{\Delta/2}) * (y^2 P_{\Delta/2}). \end{aligned}$$

Since $\|y^3 \nu\|_{\mathbb{R}} < \infty, P_{\Delta}(\mathbb{R}) = 1, \|yP_{\Delta}\|_{\mathbb{R}} \lesssim \log(1/\Delta)$ (by assumption), $\nu_{\sigma}(\mathbb{R}) < \infty$, and $\|y^2 P_{\Delta}\|_{L^1} = \mathbb{E}[L_{\Delta}^2] \lesssim \Delta$, we have that

$$\|y^3 P_{\Delta}\|_{\mathbb{R}} \lesssim \Delta \{ \|y^3 \nu\|_{\mathbb{R}} P_{\Delta}(\mathbb{R}) + \|yP_{\Delta}\|_{\mathbb{R}} \nu_{\sigma}(\mathbb{R}) \} + \|yP_{\Delta}\|_{\mathbb{R}} \|y^2 P_{\Delta}\|_{L^1} \lesssim \Delta \log(1/\Delta).$$

Now, if $\sigma > 0$, then we have that $\|yP_{\Delta}\|_{\mathbb{R}} \lesssim 1$ (cf. Remark 4.3), so that $\|y^2 P_{\Delta}\|_{\mathbb{R}} \lesssim \|yP_{\Delta}\|_{\mathbb{R}} \vee \|y^3 P_{\Delta}\|_{\mathbb{R}} \lesssim 1$. On the other hand, if $\sigma = 0$, then from (8.1), we have that

$$\|y^2 P_{\Delta}\|_{\mathbb{R}} \lesssim \Delta \|y^2 \nu\|_{\mathbb{R}} P_{\Delta}(\mathbb{R}) + \|yP_{\Delta/2}\|_{\mathbb{R}} \|yP_{\Delta/2}\|_{L^1} \lesssim \Delta + \Delta^{1/2} \log(1/\Delta) \lesssim 1,$$

where we have used that $\|yP_{\Delta/2}\|_{L^1} = \mathbf{E}[|L_{\Delta/2}|] \leq \sqrt{\mathbf{E}[L_{\Delta/2}^2]} \lesssim \Delta^{1/2}$. In either case, we have that $\|y^2P_{\Delta}\|_{\mathbb{R}} \lesssim 1$.

For y^4P_{Δ} , we have that

$$\begin{aligned} \|y^4P_{\Delta}\|_{\mathbb{R}} &\lesssim \Delta \{ \|y^4\nu\|_{\mathbb{R}}P_{\Delta}(\mathbb{R}) + \|y^3\nu\|_{\mathbb{R}}\|yP_{\Delta}\|_{L^1} + \|y^2P_{\Delta}\|_{\mathbb{R}}\nu_{\sigma}(\mathbb{R}) \} \\ &\quad + \|y^3P_{\Delta/2}\|_{\mathbb{R}}\|yP_{\Delta/2}\|_{L^1} + \|y^2P_{\Delta}\|_{\mathbb{R}}\|y^2P_{\Delta}\|_{L^1} \\ &\lesssim \Delta(1 + \Delta^{1/2}) + \Delta^{3/2}\log(1/\Delta) + \Delta \lesssim \Delta. \end{aligned}$$

Verification of $\inf_{y \in I^{\varepsilon_0}}(y^4P_{\Delta})(y) \gtrsim \Delta$ for some $\varepsilon_0 > 0$. We divide the proof into two steps.

Step 1. We first consider the case where $\sigma = 0$, $\|x\nu\|_{\mathbb{R}} < \infty$, and $\gamma_c - \int_{\mathbb{R}} x\nu(dx) = 0$ ($\|x\nu\|_{\mathbb{R}} < \infty$ ensures that $\int_{\mathbb{R}} |x|\nu(dx) < \infty$). In this case, the characteristic exponent $\psi(u)$ is

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx).$$

Observe that $\psi'(u) = i \int_{\mathbb{R}} e^{iux} x\nu(dx)$, and applying the Fourier inversion to $i^{-1}\varphi'_{\Delta} = \Delta i^{-1}\psi'\varphi_{\Delta}$, we have that $yP_{\Delta} = \Delta(y\nu) * P_{\Delta}$, so that $\|yP_{\Delta}\|_{\mathbb{R}} \leq \Delta\|y\nu\|_{\mathbb{R}}P_{\Delta}(\mathbb{R}) \lesssim \Delta$. From (8.1),

$$y^2P_{\Delta} = \Delta(y^2\nu) * P_{\Delta} + 4(yP_{\Delta/2}) * (yP_{\Delta/2}),$$

and $\|(yP_{\Delta/2}) * (yP_{\Delta/2})\|_{\mathbb{R}} \leq \|yP_{\Delta/2}\|_{\mathbb{R}}\|yP_{\Delta/2}\|_{L^1} \ll \Delta$. Since $P_{\Delta}([-\varepsilon, \varepsilon]^c) = \mathbf{P}(|L_{\Delta}| > \varepsilon) \leq \varepsilon^{-2}\mathbf{E}[L_{\Delta}^2] \lesssim \Delta$ for any $\varepsilon > 0$ by Markov's inequality, we have that $P_{\Delta}([-\varepsilon, \varepsilon]) = 1 - O(\Delta)$. So,

$$\inf_{y \in I^{\varepsilon_1/2}} ((y^2\nu) * P_{\Delta})(y) \geq P_{\Delta}([-\varepsilon_1/2, \varepsilon_1/2]) \inf_{y \in I^{\varepsilon_1}} (y^2\nu)(y) \gtrsim 1,$$

which shows that $\inf_{y \in I^{\varepsilon_1/2}}(y^2P_{\Delta})(y) \gtrsim \Delta$. This leads to the desired result with $\varepsilon_0 = \varepsilon_1/2$.

Step 2. Next, we consider the general case. Decompose the Lévy measure ν into $\nu = \nu 1_{[-\varepsilon_1/4, \varepsilon_1/4]} + \nu 1_{[-\varepsilon_1/4, \varepsilon_1/4]^c} =: \nu_1 + \nu_2$, and observe that ν_2 is a finite, non-zero measure with $\|x\nu_2\|_{\mathbb{R}} < \infty$. Then the characteristic exponent $\psi(u)$ can be decomposed as

$$\psi(u) = \underbrace{-\frac{\sigma^2 u^2}{2} + iu\gamma_1 + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\nu_1(dx)}_{=: \psi_1(u)} + \underbrace{\int_{\mathbb{R}} (e^{iux} - 1)\nu_2(dx)}_{=: \psi_2(u)},$$

where $\gamma_1 := \gamma_c - \int_{\mathbb{R}} x\nu_2(dx)$. From this decomposition, we have

$$L \stackrel{d}{=} M + N = (M_t + N_t)_{t \geq 0},$$

where $M = (M_t)_{t \geq 0}$ is a Lévy process with Lévy measure ν_1 , and $N = (N_t)_{t \geq 0}$ is a compound Poisson process with jump intensity $\nu_2(\mathbb{R})$ and jump size distribution $\nu_2/\nu_2(\mathbb{R})$ independent of M . Let Q_{Δ}, R_{Δ} denote the distributions of M_{Δ}, N_{Δ} , respectively, so that $P_{\Delta} = Q_{\Delta} * R_{\Delta}$. Since R_{Δ} is a compound Poisson distribution with absolutely continuous jump distribution, we obtain the decomposition $R_{\Delta} = R_{\Delta}(\{0\})\delta_0 + R_{\Delta}^{ac}$ where R_{Δ}^{ac} is absolutely continuous, so that $y^2P_{\Delta} = R_{\Delta}(\{0\})y^2Q_{\Delta} + y^2(R_{\Delta}^{ac} * Q_{\Delta})$. Since both y^2P_{Δ} and $R_{\Delta}^{ac} * Q_{\Delta}$ are absolutely continuous, so is y^2Q_{Δ} , and

$$(y^2P_{\Delta})(y) \geq y^2(R_{\Delta}^{ac} * Q_{\Delta})(y).$$

Now, since $y^2 R_\Delta = y^2 R_\Delta^{ac}$ and from the result of Step 1, we have that $\inf_{y \in I^{\varepsilon_1/4}} (y^2 R_\Delta^{ac})(y) \gtrsim \Delta$ and so $\inf_{y \in I^{\varepsilon_1/4}} R_\Delta^{ac}(y) \gtrsim \Delta$. Furthermore, since $Q_\Delta([-\varepsilon_1/8, \varepsilon_1/8]^c) = \mathbb{P}(|M_\Delta| > \varepsilon_1/8) \lesssim \mathbb{E}[M_\Delta^2]/\varepsilon_1^2 \lesssim \Delta$ by Markov's inequality, we have that

$$\inf_{y \in I^{\varepsilon_1/8}} (R_\Delta^{ac} * Q_\Delta)(y) \geq Q_\Delta([-\varepsilon_1/8, \varepsilon_1/8]) \inf_{y \in I^{\varepsilon_1/4}} R_\Delta^{ac}(y) \gtrsim \Delta,$$

which implies that $\inf_{y \in I^{\varepsilon_1/8}} (y^2 P_\Delta)(y) \gtrsim \Delta$. This leads to the desired result with $\varepsilon_0 = \varepsilon_1/8$. \square

8.3. Proof of Lemma 4.1. The proof is a modification of that of [56], Proposition 16 Case (iv). We will obey the notational convention used in the proof of Proposition 4.1. We first note that, under the assumption of the lemma, the Lévy measure is infinite, since for $\varepsilon \in (0, 1)$,

$$\int_{|x| \leq 1} \rho(x) dx \geq \int_{|x| \leq \varepsilon} \frac{x^2}{\varepsilon^2} \rho(x) dx \geq \frac{c}{\varepsilon},$$

and taking $\varepsilon \downarrow 0$ shows that the far left hand side is infinite. Then, Theorem 27.7 in [58] yields that P_Δ is absolutely continuous with respect to the Lebesgue measure, and by the Fourier inversion, we have that $\|y P_\Delta\|_{\mathbb{R}} \lesssim \|\varphi'_\Delta\|_{L^1}$. Observe that

$$|\varphi'_\Delta(u)| \leq \Delta \left\{ |\gamma_c| + \left| \int_{\mathbb{R}} (e^{iux} - 1) x \rho(x) dx \right| \right\} e^{-\Delta \int_{\mathbb{R}} (1 - \cos(ux)) \rho(x) dx}.$$

Since $|e^{iux} - 1| \leq |ux| \wedge 2$, we have that

$$\left| \int_{\mathbb{R}} (e^{iux} - 1) x \rho(x) dx \right| \leq |u| \int_{|x| \leq 1/|u|} x^2 \rho(x) dx + 2 \int_{|x| > 1/|u|} |x| \rho(x) dx.$$

Furthermore, since there exists a small constant $c' > 0$ such that $1 - \cos(x) \geq c'x^2$ for all $|x| \leq 1$, we have that

$$\int_{\mathbb{R}} (1 - \cos(ux)) \rho(x) dx \geq c'u^2 \int_{|x| \leq 1/|u|} x^2 \rho(x) dx.$$

From these estimates, it is seen that $\|\varphi'_\Delta 1_{[-1,1]}\|_{L^1} \lesssim \Delta$. On the other hand, since

$$c \leq |u| \int_{|x| \leq 1/|u|} x^2 \rho(x) dx \leq C \quad \text{and} \quad \int_{|x| > 1/|u|} |x| \rho(x) dx \leq C(1 + \log |u|)$$

for $|u| > 1$, we have that

$$\|\varphi'_\Delta 1_{[-1,1]^c}\|_{L^1} \lesssim \Delta \int_{[-1,1]^c} (1 + \log |u|) e^{-c'\Delta|u|} du \lesssim \log(1/\Delta),$$

where $c'' = cc'$. This completes the proof. \square

8.4. Proof of Lemma 5.1. Without loss of generality, we may assume $\alpha = 1$. Observe that

$$e^{\beta x} \int_0^\infty e^{-t - \frac{x^2}{4t}} dt = \int_0^\infty e^{-t} e^{\beta x - \frac{x^2}{4t}} dt = \int_0^\infty e^{-t(1-\beta^2)} e^{-\frac{1}{4t}(x-2\beta t)^2} dt.$$

Pick any $x, y \in \mathbb{R}$, and suppose that $(x - 2\beta t)^2 \leq (y - 2\beta t)^2$. Then

$$\begin{aligned} \left| e^{-\frac{1}{4t}(x-2\beta t)^2} - e^{-\frac{1}{4t}(y-2\beta t)^2} \right| &\leq 1 - e^{-\frac{1}{4t}\{(y-2\beta t)^2 - (x-2\beta t)^2\}} \leq \left(1 - e^{-\frac{1}{4t}\{(y-2\beta t)^2 - (x-2\beta t)^2\}} \right)^r \\ &\leq \frac{1}{4^r t^r} \{(y - 2\beta t)^2 - (x - 2\beta t)^2\}^r = \frac{1}{4^r t^r} |y - x|^r \underbrace{|y + x - 4\beta t|^r}_{\leq |y+x|^{r+4^r \beta^r t^r}} \\ &\leq \frac{1}{4^r t^r} |y - x|^r |y + x|^r + \beta^r |y - x|^r. \end{aligned}$$

By symmetry, this inequality holds for any $x, y \in \mathbb{R}$.

Now, by a change of variables, we have that $(\rho_{\#} * (h^{-1}W(\cdot/h)))(x) - \rho_{\#}(x) = \int_{\mathbb{R}} \{\rho_{\#}(x - yh) - \rho_{\#}(x)\} W(y) dy$. Observe that, since t^{-r} is integrable around the origin,

$$\begin{aligned} |\rho_{\#}(x - yh) - \rho_{\#}(x)| &\leq \frac{\delta}{\pi} \int_0^{\infty} e^{-t(1-\beta^2)} \underbrace{\left(\frac{1}{4^r t^r} |yh|^r |2x - yh|^r + \beta^r |yh|^r \right)}_{\leq 4^{-r} t^{-r} |yh|^{2r} + (2^{-r} t^{-r} |x|^r + \beta^r) |yh|^r} dt \\ &\lesssim h^{2r} |y|^{2r} + (1 + |x|^r) h^r |y|^r \end{aligned}$$

up to a constant that depends only on β, δ, r . Therefore, we conclude that

$$\|\rho_{\#} * (h^{-1}W(\cdot/h)) - \rho_{\#}\|_I \lesssim h^{2r} \int_{\mathbb{R}} |y|^{2r} |W(y)| dy + h^r \int_{\mathbb{R}} |y|^r |W(y)| dy \lesssim h^r.$$

This completes the proof. \square

9. CONVERGENCE RATES OF $\hat{\rho}$ UNDER WEIGHTED SUP-NORM ON \mathbb{R}

In this section, we study convergence rates of the spectral estimator $\hat{\rho}$ under the weighted sup-norm $\|f\|_{w,\infty} = \sup_{x \in \mathbb{R}} |x^2 f(x)|$. For this purpose, we work with conditions similar to those in Assumption 4.1.

Proposition 9.1. *Suppose that $\int_{\mathbb{R}} |x|^{2q} \rho(x) dx < \infty$ for some $q > 2$; for some $b \in \mathbb{R}$, the measure $y^4 P_{\Delta,b}(dy)$ has Lebesgue density $g_{\Delta,b}$ such that $\|g_{\Delta,b}\|_{\mathbb{R}} \lesssim \Delta$; for some $r > 0$, $x^2 \rho$ is p -times differentiable, and $(x^2 \rho)^{(p)}$ is $(r - p)$ -Hölder continuous, where p is the integer such that $p < r \leq p + 1$; let $W : \mathbb{R} \rightarrow \mathbb{R}$ be an integrable function that satisfies (4.1). Furthermore, suppose that $h \gtrsim \Delta^{1/2}$ and $\log \Delta^{-1} \lesssim \log n$. Then $\|\hat{\rho} - \rho\|_{w,\infty} = O_{\mathbb{P}}(\chi_n + h^r + h^{-1}|\hat{\sigma}^2 - \sigma^2|)$, where*

$$\chi_n = n^{-1/2} h^{-1} \log n + n^{-1} \Delta^{-1/2} h^{-1} (\log n)^2 + (n\Delta h)^{-1/2} (\log n)^{1/2} + (n\Delta)^{-1+1/q} h^{-1} \log n.$$

If in addition $\Delta \lesssim (\log n)^{-2}$ and $(n\Delta)^{-1/2+1/q} h^{-1/2} (\log n)^{1/2} \lesssim 1$, then $\chi_n \lesssim (n\Delta h)^{-1/2} (\log n)^{1/2}$.

Proof. Recall from the decomposition (7.5) that

$$\begin{aligned} x^2(\hat{\rho}(x) - \rho(x)) &= [(y^2 \rho) * (h^{-1}W(\cdot/h))](x) - x^2 \rho(x) \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (\psi''(u) - \hat{\psi}''(u)) \varphi_W(uh) du + (\sigma^2 - \hat{\sigma}^2) h^{-1} W(x/h). \end{aligned}$$

Lemma 7.9 yields that $\|[(y^2\rho) * (h^{-1}W(\cdot/h))] - x^2\rho\|_{\mathbb{R}} \lesssim h^r$. Furthermore, the expansion (7.6) holds under the assumption of the proposition, and therefore,

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} (\psi''(u) - \widehat{\psi}''(u)) \varphi_W(uh) du &= \frac{-1}{2\pi\Delta} \int_{\mathbb{R}} e^{-iux} \underbrace{\frac{(\widehat{\varphi}_\Delta - \varphi_\Delta)''}{\varphi_\Delta}}(u) \varphi_W(uh) du \\ &= \frac{1}{n\Delta h} \sum_{j=1}^n \{Y_{n,j}^2 K_n((x - Y_{n,j})/h) - \mathbb{E}[Y_{n,1}^2 K_n((x - Y_{n,1})/h)]\} \\ &\quad + O_{\mathbb{P}}\{n^{-1/2}h^{-1} \log n + n^{-1}\Delta^{-1/2}h^{-1}(\log n)^2\} \end{aligned}$$

uniformly in $x \in \mathbb{R}$. It remains to prove that

$$\begin{aligned} &\left\| \frac{1}{n\Delta h} \sum_{j=1}^n \{Y_{n,j}^2 K_n((\cdot - Y_{n,j})/h) - \mathbb{E}[Y_{n,1}^2 K_n((\cdot - Y_{n,1})/h)]\} \right\|_{\mathbb{R}} \\ &= O_{\mathbb{P}}\{(n\Delta h)^{-1/2}(\log n)^{1/2} + (n\Delta)^{-1+1/q}h^{-1} \log n\}. \end{aligned} \quad (9.1)$$

To this end, we shall apply Corollary 5.1 in [15] to the function class

$$\check{\mathcal{F}}_n = \{y \mapsto y^2 K_n((x - y)/h) : x \in \mathbb{R}\}.$$

Under the present assumption, we still have that $\|K_n\|_{\mathbb{R}} \lesssim 1$, and choose a constant $D_3 > 0$ independent of n such that $\|K_n\|_{\mathbb{R}} \leq D_3$. Let $\check{F}(y) = D_3 y^2$, which is an envelope function for $\check{\mathcal{F}}_n$. From the proof of Lemma (7.8), it is seen that K_n is of bounded variation with $\|K_n\|_{\text{TV}} \lesssim 1$, so that by Lemma 7.8 together with a simple covering number calculation, we have that for some constants $A_3, v_3 > 0$ independent of n ,

$$\sup_Q N(\check{\mathcal{F}}_n, \|\cdot\|_{Q,2}, \varepsilon \|\check{F}\|_{Q,2}) \leq (A_3/\varepsilon)^{v_3}, \quad 0 < \forall \varepsilon \leq 1.$$

Observe that $\sup_{x \in \mathbb{R}} \mathbb{E}[Y_{n,1}^4 K_n^2((x - Y_{n,1})/h)] \lesssim \Delta h$ (cf. (7.7)), and

$$\mathbb{E} \left[\max_{1 \leq j \leq n} Y_{n,j}^4 \right] \leq \left(\mathbb{E} \left[\max_{1 \leq j \leq n} |Y_{n,j}|^{2q} \right] \right)^{2/q} \leq \left(\sum_{j=1}^n \mathbb{E}[|Y_{n,j}|^{2q}] \right)^{2/q} \lesssim (n\Delta)^{2/q},$$

where we have used that $\mathbb{E}[|Y_{n,1}|^{2q}] \lesssim \Delta$, which follows from applying Theorem 1.1 in [28] with $f(x) = |x|^{2q}$. Therefore, applying Corollary 5.1 in [15] to $\check{\mathcal{F}}_n$, we conclude that

$$\mathbb{E} \left[\left\| \frac{1}{n\Delta h} \sum_{j=1}^n \{f(Y_{n,j}) - \mathbb{E}[f(Y_{n,1})]\} \right\|_{\check{\mathcal{F}}_n} \right] \lesssim \sqrt{\frac{\log n}{n\Delta h}} + \frac{\log n}{(n\Delta)^{1-1/q}h},$$

which leads to (9.1).

The last assertion is trivial, and the proof is completed. \square

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REFERENCES

- [1] Aït-Sahalia, Y. (2004). Disentangling diffusions from jumps. *J. Financial Economics* **74** 487-528.
- [2] Aït-Sahalia, Y. and Jacod, J. (2007). Volatility estimators for discretely sampled Lévy processes. *Ann. Statist.* **37** 2202-2244.
- [3] Aït-Sahalia, Y. and Jacod, J. (2014). *High-Frequency Financial Econometrics*. Princeton University Press.
- [4] Aït-Sahalia, Y. and Xiu, D. (2017). A Hausman test for the presence of noise in high frequency data. *J. Econometrics*, to appear.
- [5] Barndorff-Nielsen, O.E. and Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps. *J. Financial Econometrics* **2** 1-48.
- [6] Bickel, P. and Rosenblatt, M. (1973). On some global measures of the deviations of density function estimates. *Ann. Statist.* **1** 1071-1095. Correction (1975) **3** 1370.
- [7] Bec, M. and Lacour, C. (2015). Adaptive pointwise estimation for pure jump Lévy processes. *Stat. Inference Stoch. Process.* **18** 229-256.
- [8] Belomestny D. (2011). Statistical inference for time-changed Lévy processes via composite characteristic function estimation. *Ann. Statist.* **39** 2205-2242.
- [9] Belomestny, D. and Reiß, M. (2015). Estimation and calibration of Lévy models via Fourier methods. In: *Lévy Matters IV* (eds. D. Belomestny et al.) Springer, pp.1-76.
- [10] Bertoin, J. (1996). *Lévy Processes*. Cambridge University Press.
- [11] Bissantz, N., Dümbgen, L., Holzmann, H., and Munk, A. (2007). Non-parametric confidence bands in deconvolution density estimation. *J. Roy. Stat. Soc. Ser. B Stat. Methodol.* **69** 483-506.
- [12] Bücher, A., Hoffmann, M., Vetter, M., and Dette, H. (2017). Nonparametric tests for detecting breaks in the jump behaviour of a time-continuous process. *Bernoulli* **23** 1335-1364.
- [13] Bücher, A. and Vetter, M. (2013). Nonparametric inference on Lévy measures and copulas. *Ann. Statist.* **41** 1485-1515.
- [14] Chen, S.X., Delaigle, A., and Hall, P. (2010). Nonparametric estimation for a class of Lévy processes. *J. Econometrics* **157** 257-271.
- [15] Chernozhukov, V., Chetverikov, D., and Kato, K. (2014a). Gaussian approximation of suprema of empirical processes. *Ann. Statist.* **42** 1564-1597.
- [16] Chernozhukov, V., Chetverikov, D., and Kato, K. (2014b). Anti-concentration and honest, adaptive confidence bands. *Ann. Statist.* **42** 1787-1818.
- [17] Chernozhukov, V., Chetverikov, D., and Kato, K. (2015). Comparison and anti-concentration bounds for maxima of Gaussian random vectors. *Probab. Theory Related Fields* **162** 47-70.
- [18] Chernozhukov, V., Chetverikov, D., and Kato, K. (2016). Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related Gaussian couplings. *Stochastic Process. Appl.* **126** 3632-3651.
- [19] Chung, K.-L. (2001). *A Course in Probability Theory* (3rd edition). Academic Press.
- [20] Claeskens, G. and Van Keilegom, I. (2003). Bootstrap confidence bands for regression curves and their derivatives. *Ann. Statist.* **31** 1852-1884.
- [21] Comte, F. and Genon-Catalot, V. (2009). Nonparametric estimation for pure jump Lévy processes based on high frequency data. *Stochastic Process. Appl.* **119** 4088-4123.

- [22] Comte, F. and Genon-Catalot, V. (2010). Nonparametric adaptive estimation for pure jump Lévy processes. *Ann. Inst. H. Poincaré Probab. Stat.* **46** 595-617.
- [23] Comte, F. and Genon-Catalot, V. (2011). Estimation for Lévy processes from high frequency data within a long time interval. *Ann. Statist.* **39** 803-837.
- [24] Comte, F. and Genon-Catalot, V. (2015). Adaptive estimation for Lévy processes. In: *Lévy Matters IV* (eds. D. Belomestny et al.) Springer, pp.77-177.
- [25] Cont, R. and Tankov, P. (2004). *Financial Modeling with Jump Processes*. Chapman & Hall/CRC.
- [26] Duval, C. (2013). Density estimation for compound Poisson processes from discrete data. *Stochastic Process. Appl.* **123** 3963-3986.
- [27] van Es, B., Gugushvili, S. and Spreij, P. (2007). A kernel type nonparametric density estimator for decompounding. *Bernoulli* **13** 672-694.
- [28] Figueroa-López J.E. (2008). Small-time moment asymptotics for Lévy processes. *Statist. Probab. Lett.* **78** 3355-3365.
- [29] Figueroa-López, J.E. (2009). Nonparametric estimation for Lévy models based on discrete sampling. In: *IMS Lecture Notes of the 3rd E.L. Lehmann Symposium* **57**, pp.117-146.
- [30] Figueroa-López, J.E. (2011a). Central limit theorems for the non-parametric estimation of time-changed Lévy models. *Scand. J. Statist.* **38** 748-765.
- [31] Figueroa-López, J.E. (2011b). Sieve-based confidence interval and bands for Lévy densities. *Bernoulli* **17** 643-670.
- [32] Figueroa-López, J.E. and Houdré, C. (2009). Small-time expansions for the transition distributions of Lévy processes. *Stochastic Process. Appl.* **119** 3862-3889.
- [33] Folland, G.B. (1999). *Real Analysis* (2nd Edition). Wiley.
- [34] Giné, E. and Nickl, R. (2016). *Mathematical Foundations of Infinite-Dimensional Statistical Models*. Cambridge University Press.
- [35] Gugushvili, S. (2009). Nonparametric estimation of the characteristic triplet of a discretely observed Lévy process. *J. Nonparametric Statist.* **21** 321-343.
- [36] Gugushvili, S. (2012). Nonparametric inference for discretely sampled Lévy processes. *Ann. Inst. H. Poincaré Probab. Stat.* **48** 282-307.
- [37] Hall, P. (1991). On convergence rates of suprema. *Probab. Theory Related Fields* **89** 447-455.
- [38] Hoffmann, M. and Vetter, M. (2017). Weak convergence of the empirical truncated distribution function of the Lévy measure of an Itô semimartingale. *Stochastic Process. Appl.*, to appear.
- [39] Jacod, J. and Protter, P. (2012). *Discretization of Processes*. Springer.
- [40] Jacod, J. and Reiß, M. (2014). A remark on the rates of convergence for integrated volatility estimation in the presence of jumps. *Ann. Statist.* **42** 1131-1144.
- [41] Johannes, M. (2004). The statistical and economic role of jumps in continuous-time interest rate models. *J. Finance* **59** 227-260.
- [42] Jongbloed, G., van der Meulen, F.H., and van der Vaart, A.W. (2005). Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. *Bernoulli* **11** 759-791.
- [43] Kappus, J. (2014). Adaptive nonparametric estimation for Lévy processes observed at low frequency. *Stochastic Process. Appl.* **124** 730-758.
- [44] Kappus, J. and Reiß, M. (2010). Estimation of the characteristics of a Lévy process observed at arbitrary frequency. *Statist. Neerlandica* **64** 314-328.

- [45] Kato, K. and Sasaki, Y. (2016). Uniform confidence bands in deconvolution with unknown error distribution. arXiv:1608.02251.
- [46] Kato, K. and Sasaki, Y. (2017). Uniform confidence bands for nonparametric errors-in-variables regression. arXiv:1702.03377.
- [47] Komlós, J., Major, P., and Tusnády, G. (1975). An approximation for partial sums of independent rv's and the sample df I. *Z. Warhsch. Verw. Gabiete* **32** 111-131.
- [48] Konakov V. and Panov, V. (2016). Sup-norm convergence rates for Lévy density estimation. *Extremes* **19** 371-403.
- [49] Kou, S. (2002). A jump-diffusion model for option pricing. *Management Science* **48** 1086-1101.
- [50] Mancini, C. (2001). Disentangling the jumps of the diffusion in a geometric jumping Brownian motion. *Giornale dell'Istituto Italiano degli Attuari* **LXIV** 19-47.
- [51] McMurry, T.L. and Politis, D.N. (2004). Nonparametric regression with infinite order flat-top kernels. *J. Nonparametric Statist.* **16** 549-562.
- [52] Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *J. Financial Economics* **3** 125-144.
- [53] Neumann, M.H. and Polzehl, J. (1998). Simultaneous bootstrap confidence bands in nonparametric regression. *J. Nonparametric Statist.* **9** 307-333.
- [54] Neumann, M.H. and Reiß, M. (2009). Nonparametric estimation for Lévy processes from low-frequency observation. *Bernoulli* **15** 223-248.
- [55] Nickl, R. and Reiß, M. (2012). A Donsker theorem for Lévy measures. *J. Functional Anal.* **263** 3306-3332.
- [56] Nickl, R., Reiß, M., Söhl, J., and Trabs, M. (2016). High-frequency Donsker theorems for Lévy measures. *Probab. Theory Related Fields* **164** 61-108.
- [57] Rüschendorf, L. and Woerner, J. (2002). Expansion of transition distributions of Lévy processes in small time. *Bernoulli* **8** 81-96.
- [58] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.
- [59] Shimizu, Y. (2006). Density estimation of Lévy measures for discretely observed diffusion processes with jumps. *J. Japan Statist. Soc.* **36** 37-62.
- [60] Smirnov, N.V. (1950). On the construction of confidence regions for the density of distribution of random variables. *Doklady Akad. Nauk SSSR* **74** 189-191 (Russian).
- [61] Stein, E.M. and Weiss, G. (1971). *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press.
- [62] Trabs, M. (2015). Quantile estimation for Lévy measures. *Stochastic Process. Appl.* **125** 3484-3521.
- [63] van der Vaart, A.W. and Wellner, J.A.(1996). *Weak Convergence and Empirical Processes with Applications to Statistics*. Springer.
- [64] Vetter, M. (2014). Inference on the Lévy measure in case of noisy observations. *Statist. Probab. Lett.* **87** 125-133.
- [65] Wasserman, L. (2006). *All of Nonparametric Statistics*. Springer.

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