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April 2018

CENTER FOR RESEARCH AND EDUCATION FOR POLICY EVALUATION
DISCUSSION PAPER NO. 18

CENTER FOR RESEARCH AND EDUCATION FOR POLICY EVALUATION (CREPE)
THE UNIVERSITY OF TOKYO
http://www.crepe.e.u-tokyo.ac.jp/

FOUNDATION OF COMPETITIVE EQUILIBRIUM WITH NON-TRANSFERABLE UTILITY

IN-KOO CHO AND AKIHIKO MATSUI

Abstract. This paper investigates the dynamic foundation of a competitive equilibrium, studying a sequence of random matching models between ex ante heterogeneous buyers and sellers under two-sided incomplete information with no entry, where each agent is endowed with non-transferable utility. The economy is populated with two sets of infinitesimal agents, buyers and sellers, who have private information about their own valuations of the object. In each period, buyers and sellers in the pool are matched to draw randomly a pair of expected payoffs, which will realize if the long term relationship is formed. Each player decides whether or not to agree to form a long term relationship, conditioned on his private information. If both parties agree, then they leave the pool, receiving the expected payoff in each period while the long term relationship continues. The existing long term relationship is terminated either by will or by a random shock, upon which both parties return to the respective pools of agents. We quantify the amount of friction by the time span of each period. We demonstrate that as the friction vanishes, any sequence of stationary equilibrium outcomes, in which trade occurs with a positive probability, converges to the competitive equilibrium, under a general two sided incomplete information about the private valuation of each agent.

Keywords: Non-transferable utility, No entry, Matching, Search, Undominated equilibrium, Competitive equilibrium, Random proposal model, Single crossing property

1. Introduction

Let us consider a textbook example of a competitive market, in which agents have non-transferable utility and private information regarding their valuations of the object. Neither agents' entry into nor their exit from the economy is assumed. The market supply and demand curves intersect to determine a unique competitive equilibrium price. The goal of this paper is to provide a decentralized dynamic foundation of the textbook example of the Arrow Debreu economy, to understand whether or not and how the dispersed information can be aggregated through a decentralized trading process to achieve an efficient allocation.

A canonical model of decentralized dynamic trading can be described roughly as follows. The economy is populated by the two sets of infinitesimal agents, buyers and sellers,

Date: March 21, 2017.

We are grateful to Lones Smith and Mark Satterthwaite for their insightful remarks. Financial support from the National Science Foundation, Korea Research Foundation (2015S1A5A2A01011999) and the Scientific Research Program (Creative) of the Japan Society for the Promotion of Science (19GS0101) are gratefully acknowledged. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation or the Japan Society for the Promotion of Science.

who may or may not have private information about their own valuations of the object. Time is discrete. In each period, buyers and sellers in the pool of unmatched agents are matched and negotiate over the delivery price. If the two parties agree, then the long term relationship is formed: in each period, the seller delivers one unit of the good to the buyer at the agreed price while the relationship lasts. Friction is quantified by the duration of each period. We calculate an equilibrium of this model (with various additional elements) to see whether or not the equilibrium converges to the competitive equilibrium as friction vanishes.

To capture the key features of the textbook example of the Arrow Debreu economy, we should add three basic elements to the decentralized dynamic trading model at the same time. First, the utility function of each player is non-transferable. Second, each party may have private information about the valuation of the object, and the trading can occur under two sided incomplete information. Third, the total mass of buyers and sellers is fixed, as we assume neither agents' entry into nor their exit from the economy.

Despite a vast number of papers on the decentralized dynamic foundation of competitive equilibrium, we are not aware of any model that has all three features at the same time. Existing papers drop at least one out of the three features, in order to facilitate the analysis. Let us review the consequence of assuming each individual feature, to demonstrate how we solve the issues head on, instead of assuming away the difficult problems.

First, a significant majority of decentralized dynamic trading models are built on transferable utility with respect to transfer payment. However, there are cases in which non-transferable utility is natural. In a real estate market, for example, the amount of money one spends upon a house is so large as to affect the marginal utility of money due to income effect.

With the transferable utility function of the agent, we can invoke the powerful technique of Myerson (1981) that allows us to focus without loss of generality on the equilibrium probability of trading of a revelation game (e.g., Myerson and Satterthwaite (1983), Satterthwaite and Shneyerov (2007), Lauermann (2013) and Shneyerov and Wong (2008)). One can recover the equilibrium transfer payment from the equilibrium probability of trading and the initial condition.

Within the confines of quasi linear utility functions, Lauermann (2013) characterizes the conditions under which a sequence of stationary equilibria of decentralized trading models converges to a competitive equilibrium. That is, a sequence of stationary equilibrium outcomes of dynamic decentralized trading models converges to a Walrasian equilibrium if and only if the sequence of stationary outcomes satisfies a certain set of conditions. Lauermann (2013) then examines a number of well known examples of decentralized trading models with *transferable* utility, search cost and two sided incomplete information, to prove that his characterization result is not *vacuous*.

Because the key conditions of Lauermann (2013), such as pairwise efficiency, are built on transferable utility, the extension of Lauermann (2013) to a model with non-transferable utility is impossible, without using the equilibrium price and initial endowments. If the utility functions are not quasi linear with respect to income, the marginal utility of income depends upon price and initial endowment. Therefore, the corresponding pairwise efficiency condition for non-transferable utility involves endogenous variables such as equilibrium prices and becomes little different from the statement of Walrasian equilibrium itself, as opposed to the condition on the primitives of the model.

Also, even if the conditions of Lauermann (2013) can be extended, one still has to show that there exists a class of decentralized dynamic trading models with non-transferable utility, search cost and two sided incomplete information, in which a sequence of equilibria converges to a Walrasian equilibrium.[1] Without the existence of a convergent sequence, the characterization of a convergent sequence would be meaningless.

As we admit non-linear utility function with respect to the transfer payment, we can no longer substitute the transfer payment by a function of the probability of trading. Our key innovation is to approximate the equilibrium payoff as a function of probability of reaching agreement per period,[2] as the amount of friction vanishes. Since our focus is a dynamic trading model with little friction, this "approximation" result facilitates the analysis of the asymptotic behavior of agents as friction vanishes, in the same way as Myerson's technique does for models with transferable utility.

There are papers on decentralized trading models with non-transferable utility. Gale (1986a) and Gale (1986b) study a bilateral trading model with general utility functions. His model assumes that agents can trade as many times as possible with no cost. Since there is neither search cost nor delay cost, the agents can obtain the best bundle among the set of feasible consumption bundles, waiting for the most suitable trading opportunities for a long time. In most models in search theory, agents are faced with trade-off between good trade opportunity and quick trade. The present paper can be viewed as a first step toward the convergence result in a model with non-transferable utility and search cost.

Burdett and Wright (1998) considers a search model with non-transferable utility without the convergence result. The present model is built on their paper, including the trading protocol where the transaction price is called by the third party on which a buyer and a seller agree or not.

Green and Zhou (1998) and Kamiya and Shimizu (2006) consider search theoretic models of money with divisible money holdings. Although money is intrinsically useless object, they can compute the value function as a function of money, which exhbits concavity. If we read the value function as an indirect utility function, there is similarity between their utility function and the utility functions in the present model.

Second, in many models including Gale (1987), we often assume that the private information of each agent is revealed truthfully, immediately after the agents are matched, but before they start to negotiate the delivery price. This assumption is to avoid various problems, arising from strategic bargaining with two sided incomplete information. In particular, the same assumption ensures that the negotiation is efficient: the agreement is reached immediately, if there is a positive gain from trading.

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[1]Gale (1986a) and Gale (1986b) examined a model with non-transferable utility, assuming search cost is 0 in the sense that until agreement is reached, agents can search without incurring any cost. Under the assumption of transferable utility, Shneyerov and Wong (2008) obtained the convergence to a competitive equilibrium in a model with two sided incomplete information. Cho and Matsui (2013) obtained the convergence with non-transferable utility and search cost, but under complete information.

[2]This probability is different from what we usually refer to as the probability of delivery in Myerson (1981), which the obtained by aggregating the probability per period.

However, an important feature of the market economy is that an equilibrium is achieved even if each agent does not know the details of the transaction the agent is engaged in, in particular, the type of the opponent. Thus, if we want to understand how private information is revealed through negotiation and aggregated into the competitive equilibrium price, it seems essential to assume that the negotiation occurs under (two sided) incomplete information. The incomplete information can cause the delay in reaching agreement during bilateral negotiation (Myerson and Satterthwaite (1983)).

In contrast to the models in which the private information is revealed truthfully before negotiation (Gale (1987)), the negotiation in our model is typically inefficient: the probability of reaching agreement per period is strictly less than 1, and actually converges to 0, as friction vanishes. But, as friction vanishes, agents meet new partners more quickly, so the number of opportunities to trade increases. Even if the probability of trading per period vanishes because of two sided incomplete information, the increased opportunities to meet new partners may compensate the lost opportunities to reach agreement. The question is whether this inefficiency vanishes or not, as friction vanishes so that the frequency of meeting a new partner goes to infinity. The focus of our analysis is therefore to investigate the rate of reaching agreement and compare it to the rate of meeting a new partner.

Third, in contrast to Rubinstein and Wolinsky (1985) and Gale (1987), we fix the total mass of buyers and sellers in the economy. This modeling feature is essential, because the textbook example of the Arrow Debreu economy assumes neither entry into nor exit from the economy. As the buyers and sellers reach an agreement, they leave the pool. In order to keep the pool of unmatched buyers and sellers from being dried up, we assume that the long term relationship expires with a small probability. That is, instead of fresh agents, we "recycle" sellers and buyers. This break-up probability (also known as the layoff probability by Burdett and Wright (1998)) vanishes as friction vanishes. As pointed out earlier, the two sided private information in the negotiation causes the probability of reaching an agreement per period to vanish.

In order to prove the convergence to the competitive equilibrium, we have to show that the distribution of types of agents in the matching pool evolves in such a way that any possible gain from trading is realized through frequent interactions between buyers and sellers. Different types of agents may use different strategies, leaving the pool at different rates. Let us call a buyer "profitable" if his reservation value is greater than the corresponding Walrasian equilibrium price. Similarly, a seller is called "profitable" if her reservation value is less than the Walrasian equilibrium price. In an equilibrium, "profitable" agents reach an agreement faster than non-"profitable" agents. As the "profitable" agents leave the pool at a faster rate than non-"profitable" agents, the distribution of the pool is skewed toward non-"profitable" agents. In particular, without entry, the proportion of "profitable" agents becomes negligible, as the friction vanishes. As a result, it is not obvious whether or not "profitable" agents can be matched with each other sufficiently frequently to extract all gains from trading, which is a critical step to induce a Walrasian equilibrium.

The rest of the paper is organized as follows. We formally describe the model and the solution concept in section 2. Section 3 states the main result that follows a series of intermediate results. Section 4 discusses bilateral trading mechanisms. Section 5 concludes the paper.

2. Model

2.1. **Static environment.**

The set $I \subset \mathbb{R}$ of agents in the market are decomposed into K types of buyers, $B_1, \ldots, B_K$, and L types of sellers, $S_1, \ldots, S_L$. Each type consists of a continuum of agents. Their generic elements as well as specific ones are denoted by B_k and S_l, respectively.[3] We assume that the good is indivisible. Each seller is endowed with one unit of goods for sale. Each buyer demands up to one unit of goods, paying p.

Let $u_{bk}(d,p)$ be the utility function of type k buyer, where $d \in \{0,1\}$ and $p \geq 0$. We interpret $d = 1$ as the state in which the good is in possession of type k buyer, and $d = 0$ as the state in which the good is not in possession of type k buyer. It is assumed that u_{bk} is twice continuously differentiable. It is analytically convenient (without loss of generality) to normalize

$$u_{bk}(1,0) > u_{bk}(0,0) = 0 \qquad \forall k.$$

We need to impose a set of regularity conditions on $u_{bk}(1,p)$. We assume

$$u_{bk}(1,0) > u_{b,k+1}(1,0) \tag{2.1}$$

$$\frac{\partial u_{bk}(1,p)}{\partial p} < 0 \tag{2.2}$$

$$\frac{\partial^2 u_{bk}(1,p)}{\partial p^2} \leq 0 \tag{2.3}$$

$$\frac{\partial}{\partial p}(u_{b,k+1}(1,p) - u_{bk}(1,p)) < 0. \tag{2.4}$$

The first three properties are easy to interpret. (2.1) says that we rank the buyers according to the marginal utility of the good. (2.2) says that an increase in payment reduces the utility of the buyer, and (2.3) says that the marginal disutility of payment is increasing. The last property is the single crossing property, which implies that the marginal disutility of paying p is increasing as k increases. Combined with (2.1), (2.4) implies that

$$u_{bk}(1,p) > u_{b,k+1}(1,p) \qquad \forall k, \forall p > 0.$$

Thanks to (2.2) and (2.3), we can define the reservation value b_k implicitly as

$$u_{bk}(1,b_k) = 0$$

for each k, and (2.6) implies

$$b_1 > b_2 > \cdots > b_K.$$

We call type k buyer b_k.

[3] We assume I as a subset of $\mathbb{R}$ to simplify the exposition. To make it more general than that, we may consider a measure space $(I, \mathcal{A}, \mu)$ where $\mathcal{A}$ is a σ-algebra, and μ is a measure. We then assume the following properties: (i) $\{i\} \in \mathcal{A}$ and $\mu(\{i\}) = 0$; (ii) $\mu(I) = 2$; (iii) I is partitioned into $K + L$ measurable sets, $B_1, \ldots, B_K$, and $S_1, \ldots, S_L$; (iv) an agent $i \in B_k$ ($k = 1, \ldots, K$) is called a k type buyer, and an agent $j \in S_l$ ($l = 1, \ldots, L$) is called an l type seller.

Similarly, we define $u_{sl}(0,p)$ as the utility of type l seller when the good is delivered to the buyer, and the seller obtains p. It is assumed that u_{sl} is twice continuously differentiable. We normalize

$$u_{sl}(1,0) = 0 > u_{sl}(0,0).$$

We interpret $u_{sl}(1,0)$ as the payoff at the status quo (without trading), while $u_{sl}(0,0)$ as her payoff, if one unit of the good is sold to the buyer at price 0.

We assume that $u_{sl}(0,p)$ satisfies a set of regularity conditions.

$$u_{sl}(0,0) > u_{s,l+1}(0,0) \tag{2.5}$$

$$\frac{\partial u_{sl}(0,p)}{\partial p} > 0 \tag{2.6}$$

$$\frac{\partial^2 u_{sl}(0,p)}{\partial p^2} \leq 0 \tag{2.7}$$

$$\frac{\partial}{\partial p}(u_{s,l+1}(0,p) - u_{sl}(0,p)) < 0. \tag{2.8}$$

Define the marginal production cost s_l implicitly as

$$u_{sl}(0,s_l) = 0 \qquad \forall l.$$

We call a typical seller a type l seller or simply, an s_l seller. Under the given set of assumptions, we have

$$s_1 < s_2 < \cdots < s_L.$$

If (2.3) and (2.7) hold with equality, then the agent has the transferable utility with respect to money. As we admit (2.3) and (2.7) to hold with inequality, we allow the agents to have non-transferable utility for transfer payment.

We need these two conditions to ensure that the set of feasible payoff vectors between type k buyer and type l seller is convex. Let r be the probability that the trading occurs, and p be the payment from type k buyer to type l seller. The expected payoffs of type k buyer and type l seller are

$$\mathcal{U}_{bk}(r,p) = ru_{bk}(1,p) + (1-r)u_{bk}(0,0), \quad \text{and} \quad \mathcal{U}_{sl}(r,p) = ru_{sl}(0,p) + (1-r)u_{bk}(1,0).$$

Under (2.3) and (2.7) along with other conditions,

$$\mathcal{S}_{kl} = \{(\mathcal{U}_{bk}(r,p), \mathcal{U}_{sl}(r,p)) \mid 0 \leq r \leq 1, \quad p \geq 0, \mathcal{U}_{bk}(r,p) \geq 0, \ \mathcal{U}_{sl}(r,p) \geq 0\}$$

is compact, convex and comprehensive in $\mathbb{R}^2_+$. As long as we can ensure $\mathcal{S}_{kl}$ is compact and convex, we can replace (2.3) and (2.7) by milder conditions.

We interpret $u_{bk}(1,p)$ and $u_{sl}(0,q)$ as payoffs when the b_k buyer pays p, and s_l seller receives q, for the good delivered. To simplify notation, we write for the rest of this paper $u_{bk}(p)$ and $u_{sl}(q)$ in place of $u_{bk}(1,p)$ and $u_{sl}(0,q)$. We interpret $u_{bk}(p)$ and $u_{sl}(q)$ as the payoffs of b_k buyer and s_l seller when the good is traded, the buyer pays p, and the seller receives q. In the sequel, it is assumed that $p \geq q$ holds (no free lunch), but $p > q$ is allowed to incorporate transaction cost.

Note that if we have a buyer and a seller, the set of feasible pair of payoffs is a convex set; the Pareto frontier may not be a straight line. This occurs when there is the income effect of consumption, which can be seen in the case of houses and automobiles.

Let $x_k \in (0,\infty)$ be the measure of type k buyers and $y_l \in (0,\infty)$ be the measure of type l sellers. Let

$$X_k = \sum_{k'=1}^{k} x_{k'}$$

with $X_0 = 0$ and

$$Y_l = \sum_{l'=1}^{l} y_{l'}$$

with $Y_0 = 0$. These are exogenous variables. In order to make the model meaningful, we also assume that

$$b_1 > s_1,$$

which ensures that gains from trade exist. To simplify notation and analysis, we assume that

$$\sum_k x_k = \sum_l y_l = 1$$

holds.[4]

The demand correspondence D is a non-increasing step correspondence where

$$D(p) = \begin{cases} [X_{k-1}, X_k] & \text{if } p = b_k \\ \{X_k\} & \text{if } p \in (b_{k+1}, b_k). \end{cases}$$

Similarly, the supply correspondence S is a non-decreasing step correspondence given by

$$S(p) = \begin{cases} [Y_{l-1}, Y_l] & \text{if } p = s_l \\ \{Y_l\} & \text{if } p \in (s_l, s_{l+1}). \end{cases}$$

Generically, D and S intersect only at one point denoted by (p^*, X^*), which corresponds to a competitive equilibrium.

Let us consider the case where the vertical line of the demand curve and the horizontal line of the supply curve intersect as depicted in Figure 1 so that for appropriately chosen k^* and l^*, we have $p^* = s_{l^*}$ and $X^* = X_{k^*-1}$. Note that the marginal seller's reservation value is the market clearing price, while the marginal buyers determine the equilibrium quantity. We focus on this case as the remaining case can be analyzed in the same manner.

[4]Given the current matching technology, extending the situation to the case where the buyers' measure and the sellers' measure are not equal is straightforward. To accomodate the case of, say, $\sum_k x_k < \sum_l y_l$, we assume that $b_K = 0$ so that no trade occurs if a type K buyer meets a seller. Then those sellers who meet K type buyers can be analyzed exactly in the same manner as the one who did not meet anyone in this period.

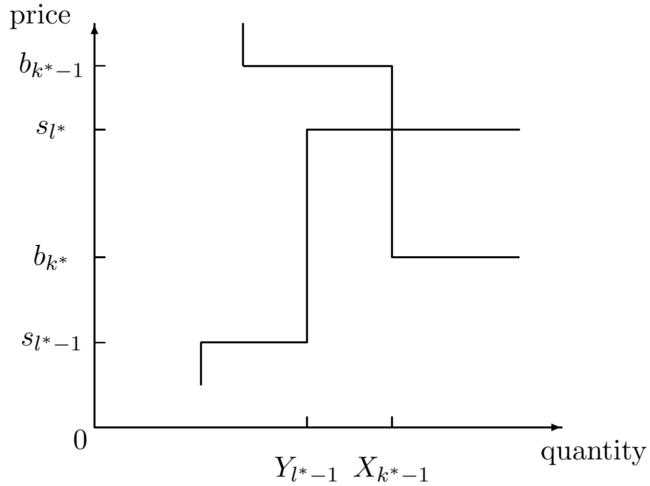

FIGURE 1. K, L

2.2. **Dynamic environment.** Time is discrete, and its generic element is written as $t \in \{1, 2, 3, \dots\}$. The time span of each period is $\Delta > 0$. In each period, there are *active* and *inactive* agents. Let z_b^k (resp. z_s^l) be the mass of type k buyers (resp. l sellers) who are active in the matching pool. Define $z = (z_b^1, \dots, z_b^K; z_s^1, \dots, z_s^L)$.

Naturally, $1 - \sum_k z_b^k$ (resp. $1 - \sum_l z_s^l$) is the mass of inactive buyers (resp. sellers). Active buyers and sellers participate in the market for transaction. There are two pools of active agents, one for the buyers and the other for the sellers. When a buyer (resp. seller) is in the pool, the buyer is randomly matched with a seller (resp. buyer) in the other pool.

Since there exists no measure space that governs a coninuum of i.i.d. random variables satisfying the law of large numbers (Feldman and Gilles (1985) and Judd (1985)), we take Boylan's approach with non-i.i.d. variables (Boylan (1992)) where the law of large numbers is assumed rather than derived. First, let B'_k and S'_l be the measurable sets of buyers and sellers in the pool, respectively, in a particular period. Write $B' = \cup_k B'_k$ and $S' = \cup_l S'_l$. Next, let ν be a measure that governs all the matches in the pool of this period. We then assume, among others, the following.

$$\begin{aligned}
\nu(\text{buyer } i \text{ meets a seller in } S'_l) &= \mu(S'_l)/\mu(S'), \\
\nu(\text{seller } j \text{ meets a buyer in } B'_k) &= \mu(B'_l)/\mu(B'), \\
\mu\left(\left\{ i \in B'_k \;\middle|\; \begin{array}{c} i \text{ meets a seller in } S'_l \\ \text{and faces price in } \quad D' \subset D \end{array} \right\}\right) &= \mu(B'_k)\mu(S'_l) \int_{D'} g^z_{kl}(p,q)d(p,q).
\end{aligned}$$

Moreover, it is assumed that these events are independent across time.[5]

We assume that one buyer is matched to one seller. Since the total mass of buyers and the total mass of sellers in the economy are equal, the size of the active buyers in the pool

[5]Another approach has been offered by Gilboa and Matsui (1992), who construct a finitely additive measure on the set of countably many agents and show that one can construct another finitely additive measure that governs the random matching with i.i.d. property of the matching where the law of large numbers is consistent with this measure.

is equal to the size of the active sellers:

$$\sum_k z_b^k = \sum_l z_s^l.$$

We assume that active buyers and sellers are randomly matched. Define

$$\mu_b^k = \frac{z_b^k}{\sum_{k'=1}^K z_b^{k'}}, \qquad \text{and} \qquad \mu_s^l = \frac{z_s^l}{\sum_{l'=1}^L z_s^{l'}}$$

as the proportion of b_k buyers and s_l sellers among buyer and sellers in the pool. An active buyer is matched to a seller with reservation value s_l with probability μ_s^l. Similarly, an active seller is matched to a buyer with reservation value b_k with probability μ_b^k.[6] We assume that the probability of a buyer (a seller) is matched to a particular type of a seller (a buyer) is independent across the pairs.

A seller can sell one unit of a good per period, and a buyer demands at most one unit of a good per period. Recall that $s_l > 0$ is the marginal cost of a seller of type $l \in \{1, \ldots, L\}$, and b_k is the marginal benefit of a buyer of type $k \in \{1, \ldots, K\}$. When a buyer and a seller meet, the two parties negotiate over the price of the good, according to which the seller delivers one unit of the good in every period indefinitely, as long as the two parties agree to continue the long term relationship. When the two parties form a partnership, the buyer agrees to pay p and the seller agrees to receive q. We assume $p \geq q$ so that there is no "free" money. We admit different prices to incorporate an inefficient bargaining which "leaves money on the table." Given a pair of proposed prices (p, q), a partnership is formed if and only if the buyer accepts p, and the seller accepts q. Then, the pair of the seller and the buyer becomes inactive by leaving the market (but not the economy). The buyer generates $u_{bk}(p)$ surplus, and the seller accrue $u_{sl}(q)$ profit per period, as long as the partnership is in effect. If the pair fails to form a partnership, each player returns to the respective pool, waiting for the next round's match.[7]

Next, suppose that a b_k buyer and an s_l seller are inactive at the beginning of period t, as they have already formed the partnership, agreeing that the buyer pays p and the

[6] Suppose that $\sum_k x_k / \sum_l y_l = \omega$. If $\omega \neq 1$, then the probability of matching is adjusted accordingly. For example, if $\omega > 1$, then a buyer is matched to a seller with a reservation value s_l with probability μ_s^l/ω, while a seller is matched to a buyer with reservation value b_k with probability μ_b^k.

[7] While our trading protocol is abstract, the trading procedure can be viewed as Chatterjee and Samuelson (1983) with a random transaction cost. Our goal is to show that a decentralized dynamic trading mechanism can achieve efficient allocation, even if the trading protocol is inefficient because of informational and institutional frictions. We chose Chatterjee and Samuelson (1983) as a benchmark, for its simplicity. In Chatterjee and Samuelson (1983), a buyer proposes p' and a seller proposes q' so that if $p' < q'$, then trading occurs and delivery price is determined as $kp' + (1-k)q'$ for some $k \in [0, 1]$. Chatterjee and Samuelson (1983) demonstrated that the presence of private information about the valuation may hinder the traders from realizing all possible gains from trading.

Suppose that each agent has to pay for the transaction cost, which is drawn from a continuous density function, which is bounded away from 0 over its domain. That is, the buyer has to pay ω_b (possibly to the mediator) in addition to $kp' + (1-k)q'$, and similarly, the seller has to pay ω_s out of the transfer payment of $kp' + (1-k)q'$. Define $p = kp' + (1-k)q' + \omega_b$ and $q = kp' + (1-k)q' - \omega_s$ as the prices a buyer and a seller has to pay, when both parties agree to trade. Since (ω_b, ω_s) is a random vector, the price each party pays may differ from each other.

seller receives q. Having already formed a partnership, neither party is active.[8] But, each party can decide whether to continue or terminate the existing partnership. If at least one party chooses to terminate the partnership, then the two players are dumped back to the respective pool of matching. If both players choose to continue the partnership, the partnership can be terminated by an exogenous shock. The probability of the exogenous shock in each period is given by $1-\delta = 1-e^{-d\Delta}$, where $d>0$. When the long term contract is dissolved by the exogenous shock, each party returns to the pool to search for a new partner. If the partnership continues, then the good is produced and delivered so that the buyer generates surplus $u_{bk}(p)$ and the seller accrues profit $u_{sl}(q)$.

We interpret $\Delta > 0$ as the amount of friction in the economy. As we are interested in the equilibrium market outcome as the friction disappears, the analysis will focus on the limit case as $\Delta \to 0$.

When a buyer b_k and a seller s_l are matched in period t, two prices p and q are proposed to the buyer and the seller randomly, and the buyer and the seller simultaneously choose whether to accept their respective prices or not. Note that the largest total surplus from trading is finite, i.e.,

$$\max_{k,l} \sup_{p,q\geq 0, p\geq q} [u_{bk}(p)+u_{sl}(q)] < \infty,$$

and

$$D = \bigcup_{k,l} \{(p,q) \mid u_{bk}(p) \geq 0 \text{ and } u_{sl}(q) \geq 0, p \geq q\}$$

is compact.

When a b_k buyer and an s_l seller are matched, the buyer reports his type as $b_{\tilde{k}}$ and the seller reports her type as $s_{\tilde{l}}$. Conditioned on the pair $(b_{\tilde{k}}, s_{\tilde{l}})$, a pair of prices $(p,q) \in D$ is drawn according to probability measure $\nu^z_{\tilde{k}\tilde{l}}$ where $z = (z_1, \ldots, z_K, z_1, \ldots, z_L)$ is the profile of measures of types of the buyers and the sellers who are in the pool. As we shall see later, we focus on the truth-telling equilibrium, i.e., $\tilde{k} = k$ and $\tilde{l} = l$. Thus, we write ν^z_{kl} in place of $\nu^z_{\tilde{k},\tilde{l}}$, which should cause no confusion.

We assume that ν^z_{kl} has a density function $g^z_{kl}(p,q)$. We impose the following regularity conditions on $g^z_{kl}(p,q)$.[9]

Assumption 2.1. *$g^z_{kl}(p,q)$ is a continuous function of (p,q,z) and satisfies*

$$g^z_{kl}(p,q) > 0 \qquad \forall p,q,z.$$

for all (p,q) with $p \geq q$ and all $z \in [0,x_1] \times \cdots \times [0,x_K] \times [0,y_1] \times \cdots \times [0,y_L]$.

In the sequel, we write ν_{kl} and $g_{kl}(p,q)$ in place of ν^z_{kl}, and $g^z_{kl}(p,q)$, respectively, whenever the meaning is clear from the context.

[8] In a certain sense, we exclude on-the-job search. However, our goal is to show that the competitive equilibrium can be sustained even if each player has little information about the economy. In fact, the ensuing analysis reveals that allowing on-the-job search only helps the convergence to the competitive equilibrium.

[9] A first time reader can assume that ν^z_{kl} is a uniform distribution over D, as the main result can be extended from the uniform distribution to a general distribution satisfying Assumption 2.1, following Cho and Matsui (2013).

Since (p, q) is drawn from D, the search process implied by ν_{kl} is not "efficient", as the trade between the two players may not extract the entire potential gain from trading. We choose to admit inefficient bargaining outcomes to understand whether or not and how an efficient outcome emerges through a decentralized trading and matching process, which may be inefficient.

One might wonder whether ν_{kl} over D can be viewed as a reduced form of a search process over a set of incentive compatible bilateral trading mechanism. In fact, our model embeds Myerson and Satterthwaite (1983) into a matching model. In order to explain that our formulation of ν_{kl} is sufficiently general to incorporate a search process over the set of incentive compatible bilateral trading mechanism, we need first to define interim expected payoff of each agent at t. Moreover, our present formulation is more convenient to explain the role of the uncertainty about the private valuation of the other players and the assumption of the infinitesimal agents in the analysis. For this reason, we will re-visit Myerson and Satterthwaite (1983) after we analyze the model completely.

Conditioned on the available information at time t, each agent chooses whether or not to agree to trade: the action space of each agent is $\{A, R\}$ where A stands for "agree" and R for "reject". If both agents agree upon (p, q), then they leave the market and become inactive. While they are out of the market, one unit of the good is delivered from the seller to the buyer, while the buyer pays p and the seller receives q in each period, as long as they are in the long term relationship. But, with probability $1 - \delta$, their partnership is terminated, and they go back to their respective pools of unmatched agents. We assume that these shocks are i.i.d. across partnerships and across time.

On the other hand, if either agent chooses R, then the buyer and seller return to their respective pools of unmatched agents, and wait for the next period for a new match. Note that even if each party agrees to the long term contract, each party is given a chance at the beginning of each period to decide whether or not to continue the long term relationship.

The timing of matches and decisions is illustrated in Figure 2.

2.3. **Histories and strategies.** We assume that in the t-th period, each agent $i \in I$ observes the proposed price $p_{i,t}$, some additional information $\theta_{i,t}$, his own action $r_{i,t}$, and the eventual fate $d_{i,t}$ of this period:

$$s_{i,t} = (p_{i,t}, \theta_{i,t}, r_{i,t}, d_{i,t}).$$

In this expression, $\theta_{i,t}$ is an additional piece of local information, i.e., the information concerning the current opponent's attributes, $r_{i,t} \in \{A, R\}$ is the reaction and $d_{i,t} \in \{0, 1\}$ is the status after $r_{i,t}$: 0 if i returns to the pool and 1 if i leaves the pool, of agent i in period t. Let Θ_i be the set of possible $\theta_{i,t}$'s. We assume that the realization of $\theta_{i,t}$ depends only on the attributes of the current opponent and oneself. For example, if agent i of type b_k meets a seller j of type s_l, then the distribution of $\theta_{i,t}$ depends only on the reported type $\tilde{s}_l$, while it may or may not contain $\tilde{s}_l$, but never contains j's past choice, say, $r_{j,t-1}$. Henceforth, we write Θ_b (resp. Θ_s) in place of Θ_i if agent i is a buyer (resp. seller).

The realized payoff $u_{i,t}$ is given by $u_{i,t} = u_{sl}(d_{i,t}, p_{i,t})$ if i is a seller of type l, $u_{i,t} = u_{bk}(d_{i,t}, p_{i,t})$ if i is a buyer of type k.[10]

[10]It is verified that $\theta_{i,t}$ is not used for decision making at all.

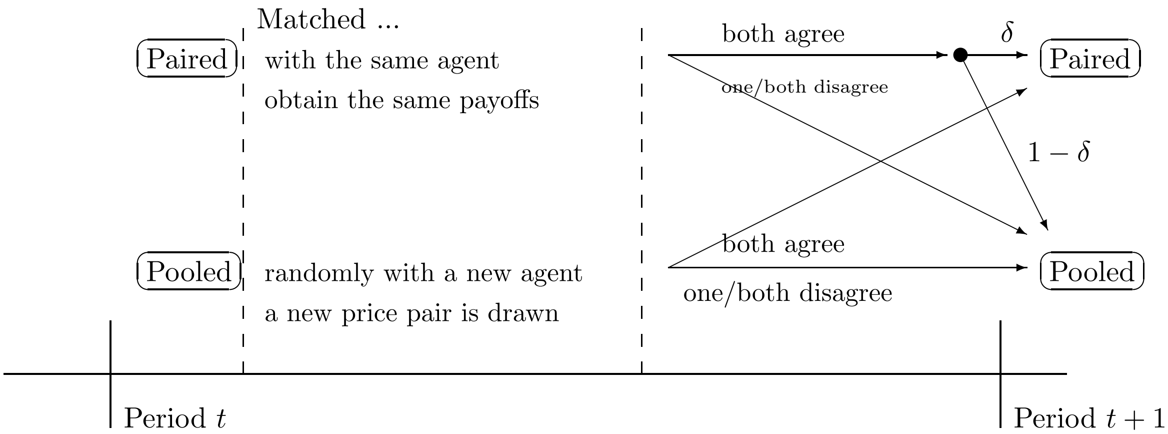

FIGURE 2. Timing of Matches and Decisions

Let $h_{i,1} = \emptyset$ be the null history. At the beginning of period $t > 1$, agent i knows

$$h_{i,t} = (s_{i,1}, \dots , s_{i,t-1})$$

which we call the private history of agent i in time t. Let $H_{i,1} = \{h_{i,1}\}$, $H_{i,t}$ $(t > 1)$ be the set of all private histories of agent i in t, and $H_i = \cup_{t \geq 1} H_{i,t}$ be the set of all private histories of agent i. Let H_i be endowed with a natural measure.[11]

Let us formalize a strategy. Let $H_{b_k} = \{h_{b_k,t}\}$ be the set of private history of buyer b_k $(k = 1, \dots , K)$. A strategy f_b^k of buyer b_k is a pair of a reporting strategy

$$\varphi_b^k : H_{b_k} \to \Delta(\Xi_b) \cup \{\emptyset\}$$

and an acceptance strategy

$$f_b^k : H_{b_k} \times [0, \infty) \times \Theta_b \to \Delta(\{A, R\}),$$

where $\Xi_b = \{1, \dots , K\}$ and $\Delta(\Xi_b)$ is the set of probability distributions over Ξ_b. Similarly, $\Delta(\{A, R\})$ is the set of probability distributions over $\Delta(\{A, R\})$. For each $h_{b_k,t-1} \in H_{b_k}$, if $d_{b_k,t-1} = 1$ so that b_k buyer is outside of the matching pool, then $\varphi_b^k(h_{b_k,t}) = \emptyset$, meaning that he has no chance to report his type to a third party. On the other hand, if $d_{b_k,t-1} = 0$ so that b_k buyer is in the matching pool, then b_k buyer reports his type according to a randomized strategy $\varphi_b^k(h_{b,t-1}) \in \Delta(\Xi_b)$. After reporting his type, b_k buyer receives a proposal $p_t \in [0, \infty)$ and θ_t, and decides whether or not to accept the proposal, conditioned on $(h_{b_k,t-1}, p_t)$. Let $\mathcal{F}_b^k$ be the set of strategies of b_k buyer, and $\mathcal{F}_b = \prod_k \mathcal{F}_b^k$.

[11]To be precise, the natural measure in this case is the product measure where the first coordinate of each $s_{i,t}$ is endowed with the Lebesgue measure, and the remaining two coordinates are endowed with the counting measures.

Similarly, we can define the strategy of s_l seller where $l \in \Xi_s = \{1, \ldots, L\}$ as a pair of a reporting strategy

$$\varphi_s^l : H_{s_l} \to \Delta(\Xi_s) \cup \{\emptyset\}$$

and an acceptance strategy

$$f_s^l : H_{s_l} \times [0, \infty) \times \Theta_s \to \Delta(\{A, R\}),$$

conditioned on the set H_{s_l} of private histories of s_l seller.

Let $\mathcal{F}_s^l$ be the set of strategies of s_l seller, and $\mathcal{F}_s = \prod_l \mathcal{F}_s^l$. Given $\mathsf{f} \in \mathcal{F} = \mathcal{F}_b \times \mathcal{F}_s$, let $\mathsf{f}_{-i} \in \mathcal{F}_{-i}$ be a strategy profile of the agents except agent i where all the other agents follow f.

A strategy profile $\mathsf{f} = (\mathsf{f}_i)_{i \in I} \in \times_{i \in I} \mathcal{F}_i$ is measurable if $\forall t$, $\forall h \in H_t$, $\forall$ open set $O_\xi \in [\Delta(\Xi_b) \cup \{\emptyset\}] \times [\Delta(\Xi_s) \times \{\emptyset\}]$, $\forall$ open set $B \subset [0, \infty)$, $\{i \in I | \; f_i(h) \in O_\xi\}$ and $\{i \in I | \; \forall p \in B[f_i(h, p) = A]\}$ are measurable. From now on, we focus on the measurable profiles of strategies.

A strategy profile $\mathsf{f} = (\mathsf{f}_i)_{i \in I}$ induces a distribution over outcome paths. In period t, a social outcome is given by

$$s_t = ((s_{i,t})_{i \in I}, \mathcal{C}_t),$$

where $\mathcal{C}_t$ is a coalitional structure at t that specifies the long term pairs and the set of unmatched agents.

Given a strategy profile $\mathsf{f} \in \mathcal{F}$, the payoff function of agent i is given by

$$\mathsf{U}_i(\mathsf{f}) = \mathsf{E}^{\mathsf{f}} \left[(1 - \beta) \sum_{t=1}^{\infty} \beta^{t-1} u_{i,t} \right] \tag{2.9}$$

where E^{f} is the expectation operator induced by f. We often omit superscript "f" to simply write "E".

2.4. **Solution concept.** We consider stationary Markov perfect equilibrium with undominated strategies. Let us define Nash equilibrium, before defining our solution concept.

Definition 2.2. *A measurable strategy profile* $\mathsf{f}^* \in \mathcal{F}$ *is a* Nash equilibrium*, if for all* $i \in I$*, for all* $\mathsf{f}_i \in \mathcal{F}_i$*,*

$$\mathsf{U}_i(\mathsf{f}^*) \geq \mathsf{U}_i(\mathsf{f}_i, \mathsf{f}^*_{-i}).$$

Given two private histories h_i, h_i', and the most recent draw p_i, define the continuation game strategy of agent i as

$$\mathsf{f}_i(h_i', p_i | h_i) = \mathsf{f}_i((h_i \circ h_i'), p_i)$$

where $h_i \circ h_i'$ is the concatenation of h_i and h_i'. Given history h, define $\mathsf{f}(\cdot | h) = (\mathsf{f}_j(\cdot | h))_{j \in I}$ as the profile of continuation strategies.

Given f, let us define the continuation value of agent i following private history $h_{i,t}$ as

$$\mathsf{U}_i(\mathsf{f} | h_{i,t}) = \mathsf{E}^{\mathsf{f}} \left[(1 - \beta) \sum_{k=0}^{\infty} \beta^k u_{i,t+k} \;\middle|\; h_{i,t} \right].$$

In an equilibrium, the continuation value of the agent in period t is a function of $d_{i,t-1}$ and $p_{i,t}$, where $d_{i,t-1} \in \{0,1\}$ indicates whether the agent is in the pool of unmatched agents ($d_{i,t-1} = 0$) or out of the pool ($d_{i,t-1} = 1$) in period $t-1$.

We are interested in a Markov perfect equilibrium with a small set of states: the one in which the strategy of agent i depends only upon the state $d_{i,t-1}$ in the previous period and the available information $(p_{i,t}, \theta_{i,t})$ of the present period. Therefore, instead of $\mathsf{U}_i(\mathsf{f}|h_i)$, we consider

$$W_{i,t}((p_{i,t}, \theta_{i,t}, r_{i,t}, 0)|\ d_{i,t-1}) = \mathsf{U}_i(\mathsf{f}|h_i \circ (p_{i,t}, \theta_{i,t}, r_{i,t}, 0))$$

and

$$W_{i,t}((p_{i,t}, \theta_{i,t}, A, 1)|d_{i,t-1}) = \mathsf{U}_i(\mathsf{f}|h_i \circ (p_i, \theta_{i,t}, A, 1))$$

for any h_i with the state of the last period being $d_{i,t-1}$.

In order to eliminate the pair of perpetual rejections from the set of stationary equilibrium outcomes, we require that agent i should accept the price p_i if and only if the conditional continuation value $W_{i,t}((p_{i,t}, \cdot, A, 1)|d_{i,t-1})$ of acceptance conditioned on the opponent's acceptance is greater than the continuation value $W_{i,t}((p_{i,t}, \cdot, \cdot, 0)|d_{i,t-1})$ of rejection.[12]

We say the outcome is *stationary* if for all t, t', all $i \in I$, all h_i with d_i' being the state of the last period, and all $(p_i, \theta_i, r_i, d_i)$,

$$W_{i,t}((p_i, \theta_i, r_i, d_i)|d_i') = W_{i,t'}((p_i, \theta_i, r_i, d_i)|d_i').$$

Since we focus on stationary outcomes, we write $W_i(\cdot)$ in place of $W_{i,t}(\cdot)$ in the sequel.

Definition 2.3. *A strategy profile* $\mathsf{f}^* = (\mathsf{f}_i^*)_{i\in I}$ *is a* stationary undominated Markov perfect (SUMP) equilibrium *if the outcome is stationary, and for all* $i \in I$*, for all* h_i *with* d_i *being the state of the last period, and all* (p_i, θ_i),

$$W_i((p_i, \theta_i, A, 1)|d_i) > W_i((p_i, \theta_i, R, 0)|d_i) \quad \Rightarrow \quad \mathsf{f}_i^*(h_i, p_i) = A,$$

and

$$W_i((p_i, \theta_i, A, 1)|d_i) < W_i((p_i, \theta_i, R, 0)|d_i) \quad \Rightarrow \quad \mathsf{f}_i^*(h_i, p_i) = R.$$

Whenever the meaning is clear from the context, we call an SUMP equilibrium an undominated equilibrium*, or simply, an* equilibrium*, for the rest of the paper.*

Note that this definition does not say anything about whether or not the opponent accepts the proposal and therefore, p_i may never be realized even if $W_i((p_i, \cdot, A, 1)|s_i) > W_i((p_i, \cdot, R, 0)|s_i)$ holds.

We have

$$W_i((p_{i,t}, \theta_{i,t}, A, 1)|d_{i,t-1}) = (1-\beta)u_i(p_{i,t}) + \beta \mathsf{E} W_i((p_{i,t+1}, \theta_{i,t+1}, r_{i,t+1}, d_{i,t+1})|d_{i,t} = 1),$$

and

$$W_i((p_{i,t}, \theta_{i,t}, r_{i,t}, 0)|d_{i,t-1}) = \beta \mathsf{E}[W_i((p_{i,t+1}, \theta_{i,t+1}, r_{i,t+1}, d_{i,t+1})|d_{i,t} = 0)],$$

Due to the nature of random matching, $\theta_{i,t}$ and $\theta_{i,t+1}$ are independent. Therefore, we have

$$W_i((p_{i,t}, \theta_{i,t}, A, 1)|d_{i,t-1}) = W_i((p_{i,t}, \theta_{i,t}', A, 1)|d_{i,t-1}),$$

[12] If the two values are equal, then the agent can either accept or reject the proposal in equilibrium.

and

$$W_i((p_{i,t}, \theta_{i,t}, r_{i,t}, 0)|d_{i,t-1}) = W_i((p_{i,t}, \theta'_{i,t}, r_{i,t}, 0)|d_{i,t-1}).$$

Thus, θ_i does not affect the continuation value, a fortiori, the decision of agent i. Moreover, only $p_{i,t}$ affects the continuation value if $d_{i,t} = 1$ holds, and none of $p_{i,t}$, $r_{i,t}$ and $d_{i,t-1}$ affect the continuation value if $d_{i,t} = 0$ holds. Hence, we can write

$$W_i(p_{i,t}) = W_i((p_{i,t}, \theta_{i,t}, A, 1)|d_{i,t-1})$$

and

$$W_i = W_i((p_{i,t}, \theta_{i,t}, r_{i,t}, 0)|d_{i,t-1}).$$

It is straightforward to show the existence of an undominated equilibrium.

Theorem 2.4. *Under Assumption 2.1, an undominated equilibrium exists.*

Proof. See Appendix A. □

Given a profile φ of the equilibrium reporting strategies, let $\hat{\nu}_{\varphi_b^k, \varphi_s^l}$ be the equilibrium distribution which selects (p, q) accordingly. Following the spirit of the revelation principle, consider a mapping

$$(b_k, s_l) \mapsto \hat{\nu}_{\varphi_b^k, \varphi_s^l}$$

and define

$$\nu_{kl} = \hat{\nu}_{\varphi_b^k, \varphi_s^l}$$

which is induced by a density function

$$g_{kl}^{\nu}(p, q) = \sum_{k'} \sum_{l'} g_{k'l'}^{\hat{\nu}}(p, q) \varphi_b^k(k') \varphi_s^l(l').$$

Note that g_{kl}^{ν} satisfies Assumption 2.1.

Since the equilibrium strategy is conditioned on the private type of each player rather than the identity of the player, it is more convenient to write W_b^k as the expected payoff of b_k buyer, and W_s^l as the expected payoff of s_l seller, instead of W_i. The equilibrium is determined only by the continuation game payoff. Given p, for example, buyer b_k must make the same decision as long as the continuation game payoff is the same, even if (p, q) is drawn from a different distribution or is realized by a different reporting strategy. Thus, W_b^k is unaffected by φ_b^k or φ_s^l, as long as the reporting strategy induces the same ν_{kl}. Hence, we can regard ν_{kl} instead of $\hat{\nu}_{\varphi_b^k, \varphi_s^l}$ as the equilibrium outcome, in which each player reports his type truthfully.

2.5. **Optimal response and value function.** The optimal strategy of b_k buyer is to accept p if $W_b^k(p) > W_b^k$ and reject it if $W_b^k(p) < W_b^k$. We have

$$W_b^k(p) = (1 - \beta) u_{bk}(p) + \beta \left(\delta W_b^k(p) + (1 - \delta) W_b^k \right).$$

Thus, it is verified that the optimal strategy of b_k buyer is to accept p if

$$u_{bk}(p) > W_b^k$$

and reject p if

$$u_{bk}(p) < W_b^k.$$

Similarly, the optimal strategy of s_l seller is to accept q if

$$u_{sl}(q) > W_s^l$$

and reject q if

$$u_{sl}(q) < W_s^l.$$

Define equilibrium threshold p_k of b_k buyer implicitly as

$$u_{bk}(p_k) = W_b^k$$

and similarly,

$$u_{sl}(q_l) = W_s^l$$

as the equilibrium threshold of s_l seller.

Suppose that $(p,q) \in D$ is drawn according to ν_{kl}. Define

$$\Pi_{kl} = \{(p,q) \in D \mid u_{bk}(p) \geq W_b^k, \quad u_{sl}(q) \geq W_s^l\}$$

as the pair of prices which are accepted by both parties, and

$$\pi_{kl} = \nu_{kl}(\Pi_{kl})$$

as the probability of such an event. Recall that $z_b^k(z_s^l)$ is the mass of type k buyers (respectively, l sellers) in the pool. In a stationary equilibrium, the size of the pool must remain constant, balancing the inflow and the outflow of agents from the pool of singles.

$$z_b^k \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'} = (x_k - z_b^k)(1-\delta), \tag{2.10}$$

$$z_s^l \sum_{k'=1}^{K} \mu_b^{k'} \pi_{k'l} = (y_l - z_s^l)(1-\delta). \tag{2.11}$$

We have

$$W_b^k(p) = (1-\beta)u_{bk}(p) + \beta\left(\delta W_b^k(p) + (1-\delta)W_b^k\right)$$

and

$$W_b^k = (1-\beta)0 + \beta\left((1 - \sum_{l=1}^{L} \mu_s^l \pi_{kl})W_b^k + \sum_{l=1}^{L} \mu_s^l \pi_{kl} \mathsf{E}(W_b^k(p)|\Pi_{kl})\right).$$

Similarly,

$$W_s^l(q) = (1-\beta)u_{sl}(q) + \beta\left(\delta W_s^l(q) + (1-\delta)W_s^l\right)$$

and

$$W_s^l = (1-\beta)0 + \beta\left((1 - \sum_{k=1}^{K} \mu_b^k \pi_{kl})W_s^l + \sum_{k=1}^{K} \mu_b^k \pi_{kl} \mathsf{E}(W_s^l(q)|\Pi_{kl})\right).$$

where $W_s^l(q)$ is the value function of s_l seller who is in the long term relationship with a buyer, by agreeing to receive q. After substituting $W_b^k(p)$ and $W_s^l(q)$ into W_b^k and W_s^l, respectively, we have

$$W_b^k = \frac{\beta \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'} \mathsf{E}(u_{bk}(p)|\Pi_{kl'})}{1-\beta\delta+\beta\sum_{l'=1}^{L}\mu_s^{l'}\pi_{kl'}} \tag{2.12}$$

$$W_s^l = \frac{\beta \sum_{k'=1}^{K} \mu_b^{k'} \pi_{k'l} \mathsf{E}(u_{sl}(q)|\Pi_{k'l})}{1-\beta\delta+\beta\sum_{k'=1}^{K}\mu_b^{k'}\pi_{k'l}} \tag{2.13}$$

Note that the present payoff without agreement is normalized to 0. After moving terms, we can write

$$W_b^k = \frac{\beta \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'} \mathsf{E}(u_{bk}(p) - W_b^k|\Pi_{kl'})}{1-\beta\delta} \tag{2.14}$$

$$W_s^l = \frac{\beta \sum_{k'=1}^{K} \mu_b^{k'} \pi_{k'l} \mathsf{E}(u_{sl}(q) - W_s^l|\Pi_{k'l})}{1-\beta\delta}. \tag{2.15}$$

3. Analysis

3.1. **Overview.** As depicted in Figure 1, the competitive equilibrium price is determined by s_{ℓ^*} sellers. Let us call a s_{ℓ^*} seller the marginal type, that determines the market clearing price. In the competitive equilibrium, a positive fraction of s_{ℓ^*} sellers may not trade, while all the sellers whose production cost is less than s_{ℓ^*} trade, and all buyers whose marginal utility is higher than s_{ℓ^*} trade in the competitive equilibrium. Being the marginal type, the surplus of s_{ℓ^*} sellers in the competitive equilibrium is equal to 0. The main thrust of the analysis is to show that the behavior of the marginal type, s_{ℓ^*}, in the decentralized trading model emulates the behavior of the corresponding type in the competitive equilibrium.

The main challenge of the analysis is to prove so in the decentralized dynamic trading model. Being the marginal type, the equilibrium payoff of s_{ℓ^*} sellers should be zero in the limit, as friction vanishes. But, along the convergent sequence of equilibria of the present decentralized trading model, the equilibrium payoff of s_{ℓ^*} sellers should be strictly positive.[13]

One may wonder at this stage why we cannot directly show all (relevant) thresholds converge to the same value. π_{kl} is determined by the difference of the equilibrium thresholds of b_k buyer and s_l seller. If we can show that $\pi_{kl} \to 0$ as $\Delta \to 0$ for all k and all l with $b_k > s_l$, then it follows that all equilibrium thresholds converge to a single price, from which we can recover the law of single price and move on to prove the convergence to the competitive equilibrium.

The conventional models assume efficient bargaining protocol and free entry of fresh agents, in combination with transferable utility. Transferable utility allows us to focus on the probability of delivery without loss of generality, which greatly simplifies the analysis.

[13] The statements in this paragraph do not imply that it is trivial to show that other types' behavior in the decentralized model emulates their behavior in the competitive equilibrium. We check their behavior one by one, which is the reason that there are many lemmata in the following subsections other than the ones mentioned in the next paragraph.

Thanks to free entry of fresh agents, all μ_b^k's and μ_s^l's were bounded away from zero (Gale (1987) and Mortensen and Wright (2002)). Then, one can prove that all (relevant) thresholds converge to the same value.

In our case, however, we cannot invoke Myerson (1981) to substitute the transfer payment as a function of probability of reaching agreement. Without fresh entry of agents into the economy, the size of pool becomes smaller, as friction vanishes. Since buyers with high reservation values and sellers with low reservation values are matched away from their respective pools more quickly than others, their fractions in the matching pool converge to zero. Therefore, for k type buyers with $\mu_b^k \to 0$ as $\Delta \to 0$, we need to show that $\mu_b^k/(1-\beta\delta) > 0$ holds as $\Delta \to 0$ in order to show that $\pi_{kl} \to 0$ as $\Delta \to 0$ for all (relevant) l's. As we need to show the "profitable" types vanish, but do so sufficiently slowly, the actual proof is significantly more complicated than one might presume.

The core exercise is to prove five intermediate results. The first result is to address the issues arising from non-transferable utility. The rest is to handle the challenges arising from both non-transferable utility and no free entry.

First, we demonstrate in Lemma 3.1 that the transfer payment, and therefore, the long run average equilibrium payoff, can be approximated as a function of probability of reaching agreement *per period*,[14] asymptotically. In contrast to Myerson (1981), where we can substitute the transfer payment *exactly* by a function of the probability of reaching agreement, our result holds only in the limit. As a result, the equilibrium payoff is sensitive to the rate at which the probability of reaching agreement *per period* vanishes in the limit.

Second, the equilibrium payoff of s_{ℓ^*} type seller vanishes at a "right" rate to ensure that all sellers whose marginal production cost is less than s_{ℓ^*} leave the market with probability one (Lemma 3.6). Third, a positive fraction of s_{ℓ^*} sellers remain in the pool (Lemma 3.7). Fourth, the equilibrium payoff of s_{ℓ^*} sellers vanishes in the limit, while it remains positive along the convergent sequence of equilibria of the decentralized dynamic trading models (Lemma 3.8). Fifth, the proportion of any buyer whose marginal utility is higher than the marginal cost s_{ℓ^*} must vanish from the pool in the limit (Lemma 3.13). The convergence to the competitive equilibrium (Theorem 3.14) follows, among others, from these intermediate results.

3.2. **Preliminaries.** Note that for each k and l, if ν_{kl} is a uniform distribution, then π_{kl} is proportional to the size of the area of Π_{kl}. For a general distribution satisfying Assumption 2.1, we have the following lemma.

Lemma 3.1. *Given k and l, there exist $\underline{A}_b, \overline{A}_b, \underline{A}_s, \overline{A}_s > 0$ and $\alpha > 0$ such that*

$$\frac{\beta \underline{A}_b \sum_{l=1}^{L} \mu_s^l (\pi_{kl})^{1+\alpha}}{1-\beta\delta} \leq W_b^k \leq \frac{\beta \overline{A}_b \sum_{l=1}^{L} \mu_s^l (\pi_{kl})^{1+\alpha}}{1-\beta\delta}, \tag{3.16}$$

$$\frac{\beta \underline{A}_s \sum_{k=1}^{K} \mu_b^k (\pi_{kl})^{1+\alpha}}{1-\beta\delta} \leq W_s^l \leq \frac{\beta \overline{A}_s \sum_{k=1}^{K} \mu_b^k (\pi_{kl})^{1+\alpha}}{1-\beta\delta}. \tag{3.17}$$

In particular, if ν is the uniform distribution over D, then $\alpha = 1/2$.

Proof. See Appendix B. □

[14] This is a different objective from what we consider in the mechanism design approach, where we use the probability reaching agreement in the *entire* game rather than per period.

Lemma 3.1 suppresses the transfer payment from the analysis, in the same way as the mechanism design approaches does under the assumption of the quasi linear utility function. The substance of the lemma is that the expected gain from reaching a long term agreement is approximated by a monotonic function of the probability of reach agreement. We do not need the quasi linear utility function, but need only continuity of the utility function and continuity of the density function of the probability distribution of (p, q). For the rest of the paper, we focus on the asymptotic properties of π_{kl}, which hold the clue of the asymptotic properties of the value functions.

The only property we need is that the value function converges to zero as Δ goes to zero at a faster rate than π_{kl}'s: $\exists \alpha > 0$ such that the value function vanishes at the rate of $\pi_{kl}^{1+\alpha}$. Existence of such $\alpha > 0$ is guaranteed by ν_{kl}, which has a continuous density, uniformly bounded away from 0 over its support.

Recall that

$$u_{bk}(p_k) = W_b^k \tag{3.18}$$

is the threshold price of b_k buyer, and

$$u_{sl}(q_l) = W_s^l \tag{3.19}$$

is the threshold price of s_l seller. We can show that the equilibrium thresholds are monotonic with respect to the marginal valuation or cost.

Lemma 3.2. *For all k and l, we have*

$$p_k \geq p_{k+1} \quad \textit{and} \quad q_l \leq q_{l+1}.$$

Proof. See Appendix C. □

Since the threshold is an increasing function of the types, we have a natural monotonic relationship among the probabilities of reaching agreement between the two agents.

Lemma 3.3.

$$\forall k \forall l \ [(\pi_{kl} = 0) \ \rightarrow \ (\pi_{k+1,l} = 0) \ \textit{and} \ (\pi_{k,l+1} = 0)] \tag{3.20}$$

Proof. See Appendix D. □

In most existing models, a positive mass of entrants ensures that the matching pool maintains a positive mass of agents who can obtain positive gain from trading, even in the limit as $\Delta \to 0$ (e.g., Rubinstein and Wolinsky (1985)). In contrast, agents are entering the matching pool only when the existing long term relation dissolves. As $\Delta \to 0$, the mass of entrants into the pool vanishes, and so does the mass of agents in the pool who can gain positive surplus from the long term relationship.

The stationary distribution of types of the agents in the pool is skewed so that the pool is inundated by the buyers with low reservation values and the sellers with high production costs. As a result, a seller with low production cost, for example, may have to go through many rounds of matching in the pool before making a profitable long term relationship. As $\Delta \to 0$, a seller in the pool has increasingly many opportunities to be matched to a buyer. We need to examine whether it takes a positive amount of real time before forming a profitable long term relationship in an undominated stationary equilibrium. The core of the proof is essentially to show that the real time necessary to find a profitable long term relationship in the pool also vanishes as $\Delta \to 0$.

We first explore the properties of endogenous variables which affect the buyer's equilibrium behavior, and then examine the other variables affecting the seller's equilibrium behavior. The main result obtains by combining these results, which also provide a useful insight into the structure of an equilibrium.

3.3. **Buyers.** We focus on the case in which the competitive equilibrium price is equal to s_{l^*}, and the equilibrium quantity is X_{k^*-1}. We first establish a weaker version of this statement in the context of the matching model, saying that b_{k^*} buyer cannot reach agreement with any seller, or s_{l^*} seller cannot trade with any buyer.

Lemma 3.4. *$\exists \overline{\Delta} > 0$ such that $\forall \Delta < \overline{\Delta}$, $[\forall l(\pi_{k^*l} = 0)]$ or $[\forall k(\pi_{kl^*} = 0)]$.*

Proof. See Appendix E. □

Using the fact that

$$Y_{l^*-1} < X_{k^*-1},$$

we strengthen Lemma 3.4. That is, b_{k^*} buyer does not trade with any seller, and there exists some type b_k of the buyer who will trade with s_{l^*} seller with a positive probability.

Lemma 3.5. *$\exists \overline{\Delta} > 0$, $\forall \Delta < \overline{\Delta}$, $\forall l(\pi_{k^*l} = 0)$ and $\exists k(\pi_{kl^*} > 0)$.*

Proof. See Appendix F. □

Lemma 3.4 is weaker than what we need, as it only says that there exists k such that $\pi_{kl^*} > 0$. We need to show that $\pi_{k^*-1,l^*} > 0$, which implies that $\forall k < k^*$, $\pi_{kl^*} > 0$ $\forall \Delta > 0$. To this end, we need to investigate the properties of a seller's equilibrium strategy.

3.4. **Sellers.** If a sequence of undominated stationary equilibria of a matching model converges to the competitive equilibrium, then s_l seller with $s_l < s_{l^*}$ should be matched away almost immediately, in order to ensure the efficiency of the allocation. This is a crucial property for obtaining the ex post efficiency of the equilibrium allocation.

However, proving this property for a matching model is not as straightforward as the competitive equilibrium model suggests, because the matching between the two parties is not conditioned on the reservation value of either party. Moreover, even if $s_l \geq s_{l^*}$, s_l seller may trade with a higher valuation buyer than b_{k^*-1}. The substance of the next lemma is to show that the low reservation value seller must be matched away almost immediately in the steady state, in order to exhaust possible gains from trading.

Lemma 3.6. *$\forall l < l^*$ $\lim_{\Delta \to 0} z_s^l = 0$.*

Proof. See Appendix G. □

Note that Lemma 3.6 does not imply

$$\forall l < l^* \ \lim_{\Delta \to 0} \mu_s^l = 0,$$

unless we prove that the size of the pool is bounded away from 0. Combining Lemma 3.6 with Lemma 3.5, the next lemma shows that the matching pool does not shrink away even in the limit, as friction vanishes. In particular, a positive portion of s_{l^*} seller must be in the matching pool.

Lemma 3.7. $\liminf_{\Delta\to 0} z_s^{l^*} > 0$ *and* $\lim_{\Delta\to 0}\mu_s^l = 0$ *for all* $l < l^*$.

Proof. See Appendix H. □

In a competitive equilibrium model, if $s_l > s_{l^*}$, then s_l seller cannot trade profitably and her surplus is 0, and if $s_l < s_{l^*}$, s_l seller receives positive surplus from trading. The same model asserts that if

$$Y_{l^*-1} < X_{k^*-1},$$

then s_{l^*} seller receives 0 surplus.

This seemingly obvious observation in a competitive equilibrium is no longer obvious in a matching model. If s_{l^*} seller trades with buyers with a positive probability according to Lemma 3.5, then her long run average payoff is not 0. The next lemma reconciles this discrepancy, by showing that as friction vanishes, the surplus of s_{l^*} seller remains positive, but converges to 0.

Lemma 3.8. $\forall\Delta > 0$, $W_s^{l^*} > 0$ *and*

$$\lim_{\Delta\to 0} W_s^{l^*} = 0.$$

Proof. See Appendix I. □

3.5. **Buyers and sellers.** In a competitive equilibrium, whenever $b_k(\geq b_{k^*-1})$ buyer trades with a seller s_{l^*}, then the trading between the two players must be efficient in the sense that it exhausts all possible gains from trade. The next lemma shows that the bilateral trading between $b_k(\geq b_{k^*-1})$ buyer and s_{l^*} seller must be asymptotically efficient.

Lemma 3.9. $\forall k < k^*$ $\lim_{\Delta\to 0}\pi_{kl^*} = 0$.

Proof. See Appendix J. □

Lemma 3.9 is weaker than what we need, as it admits the possibility that $\pi_{kl^*} = 0$ along the sequence as $\Delta \to 0$. We need to exclude this possibility, because $\pi_{kl^*} = 0$ implies that b_k buyer does not trade with s_{l^*} seller even if a positive gain from trading exists.

In order to strengthen Lemma 3.9, we prove a series of intermediate results. Using Lemma 3.9, we show that if $s_l > s_{l^*}$, then s_l seller cannot trade with any buyer with a positive probability.

Lemma 3.10. $\exists\bar{\Delta}\forall\Delta \in (0,\bar{\Delta})$ $[\forall l > l^*\ \forall k\ \pi_{kl} = 0]$.

Proof. See Appendix K. □

We prove that if $b_k > b_{k^*}$, then b_k buyer's equilibrium payoff must be positive even in the limit as $\Delta \to 0$.

Lemma 3.11. $\forall k < k^*$ $\lim_{\Delta\to 0} W_b^k > 0$.

Proof. See Appendix L. □

As in Lemma 3.6, if $b_k > b_{k^*}$, then the proportion of b_k buyer vanishes from the matching pool, as friction disappears.

Lemma 3.12. $\forall k < k^*$ $\lim_{\Delta\to 0} z_s^k = 0$.

Proof. See Appendix M □

We now prove a stronger version of Lemma 3.9.

Lemma 3.13. $\forall k < k^*$, $\forall\{\Delta\}$ *converging to 0,* $\pi_{kl^*} > 0$ *except for at most finitely many number of* Δ*'s, and* $\lim_{\Delta\to 0}\pi_{kl^*} = 0$.

Proof. See Appendix N □

The main result says that all transactions will be made around the competitive equilibrium price p^*, and that rationing occurs properly in the sense that those who are supposed to trade in the competitive equilibrium actually leave the pool almost always in the limit. We state the proof to show how the intermediate results are used to prove the main result. Let p_k be the equilibrium threshold price of b_k buyer and q_l be the equilibrium threshold price of s_l seller.

Theorem 3.14. $\forall k < k^*$ $\forall l \leq l^*$ $\lim_{\Delta\to 0} p_k = \lim_{\Delta\to 0} q_l = s_{l^*} = p^*$ *which is the market clearing price. Moreover,* $\forall k < k^*$ $\forall l < l^*$ $\lim_{\Delta\to 0} z_b^k = \lim_{\Delta\to 0} z_s^l = 0$.

Proof. By Lemma 3.13, $\pi_{kl^*} > 0$ $\forall\Delta > 0$. Thus,

$$p_k \geq p_{k^*-1} \geq q_{l^*} \qquad \forall\Delta > 0.$$

By Lemma 3.13, $\lim_{\Delta\to 0}\pi_{kl^*} = 0$ $\forall k < k^*$. Hence,

$$\lim_{\Delta\to 0} p_k - q_{l^*} = 0.$$

Since $\lim_{\Delta\to 0} W_s^{l^*} = 0$ by Lemma 3.8,

$$\lim_{\Delta\to 0} p_k = s_{l^*}.$$

Next, we show

$$\lim_{\Delta\to 0} q_l = s_{l^*} \qquad \forall l \leq l^*.$$

Suppose the contrary: $\exists l \leq l^*$ such that

$$\limsup_{\Delta\to 0} q_l > s_{l^*}.$$

Then, $\exists\overline{\Delta}$ such that $\forall\Delta \in (0, \overline{\Delta})$, $W_s^l = 0$.

By Lemma 3.8, $W_s^{l^*} > 0$ $\forall\Delta > 0$, even though $\lim_{\Delta\to 0} W_s^{l^*} = 0$. Since $s_l < s_{l^*}$, $W_s^l \geq W_s^{l^*} > 0$. Thus, $W_s^l = 0$ for some $\Delta > 0$ is impossible.

Suppose $\liminf_{\Delta\to 0} q_l < s_{l^*}$. Then one can find k such that $\pi_{kl} > 0$ for a sufficiently small $\Delta > 0$. Then we can repeat the same exercise to draw a contradiction. Thus, we have

$$\lim_{\Delta\to 0} q_l = s_{l^*},$$

as desired.

The second statement of the theorem is the direct consequence of Lemma 3.7 and Lemma 3.13. □

4. Bilateral Trading Mechanisms

We formulate the trading protocol as a random search process for a pair of prices (p, q) which is drawn according to probability distribution ν_{lk}. As the trading occurs possibly under incomplete information, one might wonder whether our formulation is sufficiently general to incorporate a large class of incentive compatible trading mechanisms. Indeed, the randomly drawn pair of prices for b_k seller and s_l seller can be viewed as a randomly drawn pair of equilibrium payoff vectors associated with an incentive compatible trading mechanism.

To see this, let us assume a trading model endowed with quasi linear utility function, as most existing papers in the literature assume. Imagine that when a buyer and a seller are matched in the pool, the two players randomly search for an incentive compatible trading mechanism, based upon the information available at the time of matching. Since we focus on a stationary equilibrium, let us drop time subscripts from the variables in order to simplify notation.

Let $g(b, s)$ be the probability distribution of the types of the buyers and the sellers in the pool in an undominated stationary equilibrium. Define $g_b(s|b_k)$ and $g_s(b|s_l)$ as the distributions over the seller's type and the buyer's type conditioned on b_k and s_l, respectively. Let $(y(b_k, s_l), x(b_k, s_l))$ be the pair of the delivery price and the probability of delivery conditioned on the reported types (b_k, s_l) of a buyer and a seller.

We say that (y, x) is short run incentive compatible, if

$$\begin{aligned}
\mathcal{U}_b^k(b_k) &= \int_s (x(b_k, s)b_k - y(b_k, s))\, g_b(s|b_k)ds \\
&\geq \int (x(b_{k'}, s)b - y(b_{k'}, s))\, f_b(s|b_k)ds = \mathcal{U}_b^k(b_{k'}) \qquad \forall k, k' \\
\mathcal{U}_s^l(s_l) &= \int_b (y(b, s_l) - x(b, s_l)s_l)\, g_s(b|s_l)db \\
&\geq \int (y(b, s_{l'}) - x(b, s_{l'})s_l)\, f_s(b|s_l)db = \mathcal{U}_s^l(s_{l'}) \qquad \forall l, l',
\end{aligned}$$

where $\mathcal{U}_b(b_k)$ and $\mathcal{U}_s(s_l)$ are the interim expected utility per period of b_k buyer and s_l seller during the long term relationship.

Let $\mathcal{M}(g)$ be the set of all short run incentive compatible mechanisms for a given distribution g. We know that $\mathcal{M}(g)$ is not empty, because any mechanism with a constant delivery price is incentive compatible. Since the incentive constraint is a linear constraint on the set of feasible mechanisms, $\mathcal{M}(g)$ is convex and compact. At this moment, we do not consider individual rationality, and some element in $\mathcal{M}(g)$ may generate a negative expected one shot payoff to some type of a player.

The objective function of each agent is the long run average discounted payoff. Thus, we need to spell out the incentive compatibility condition in terms of the long run average discounted payoff rather than short run payoff functions, $\mathcal{U}_b^k$ or $\mathcal{U}_s^l$.

Abusing notation, let us write $\mathcal{W}_b^k(b_k, s_l)$ and $\mathcal{W}_s^l(b_k, s_l)$ as the continuation value of b_k buyer and s_l seller, conditioned on the two players agreeing on (y, x), and reporting truthfully their types. A simple calculation shows that

$$\begin{aligned}\mathcal{W}_b^k(b_k, s_l) &= \frac{(1-\beta)(y(b_k,s_l)b_k - x(b_k,s_l)) + \beta(1-\delta)\mathcal{W}_b^k}{1-\beta\delta} \\ \mathcal{W}_s^l(b_k, s_l) &= \frac{(1-\beta)(x(b_k,s_l) - y(b_k,s_l)s_l)) + \beta(1-\delta)\mathcal{W}_b^k}{1-\beta\delta}\end{aligned}$$

We say that (y, x) is (long run) incentive compatible, if

$$\begin{aligned}\mathcal{W}_b^k(b_k) = \int_s \mathcal{W}_b^k(b_k, s) g_b(s|b_k) ds &\geq \int_s \mathcal{W}_b^k(b_{k'}, s) g_b(s|b_k) ds \qquad \forall k, k' \\ \mathcal{W}_s^l(s_l) = \int_b \mathcal{W}_s^l(b, s_l) g_s(b|s_l) db &\geq \int_b \mathcal{W}_s^l(b, s_{l'}) g_s(b|s_l) db \qquad \forall l, l'.\end{aligned}$$

Let $\mathcal{W}_b^k(b_{k'})$ be the interim expected long run discounted average payoff of b_k buyer if he reports his type as $b_{k'}$ instead. Note that $\mathcal{W}_b^k(b_k)$ is a linear function of $\mathcal{U}_b^k(b_k)$. Given $(W_b^1, \ldots, W_b^K; W_s^1, \ldots, W_s^L)$,

$$\mathcal{W}_b^k(b_k) \geq \mathcal{W}_b^k(b_{k'})$$

if and only if

$$\mathcal{U}_b^k(b_k) \geq \mathcal{U}_b^k(b_{k'})$$

where the inequality is implied by the incentive compatibility of (y, x). Thus, the short run incentive compatibility implies the incentive compatibility, which implies that $\forall (y, x) \in \mathcal{M}(g)$, each agent has incentive to report his type truthfully, for given distribution g.

In any equilibrium, the optimal decision of b_k buyer is therefore to accept (y, x) if

$$\mathcal{U}_b^k(b_k) > W_b^k$$

and reject

$$\mathcal{U}_b^k(b_k) < W_b^k.$$

The optimal decision of s_l seller can be written in the same manner.

Note that in any undominated equilibrium, a buyer makes a decision whether to accept or reject (y, x), conditioned on the event that (y, x) is accepted by a seller. Thus, $g_b(s_l|b_k)$ must be consistent with the optimal decision of a seller. Similarly, $g_s(b_k|s_l)$ must be consistent with the optimal decision of a buyer.

Let $\tilde{g}(b_k, s_l)$ be the density of (b_k, s_l) in the pool, and $\hat{g}(b_k, s_l)$ be the probability that (b_k, s_l) form a long term relationship. Then,

$$g(b_k, s_l) = \frac{\tilde{g}(b_k, s_l)\hat{g}(b_k, s_l)}{\sum_{k', l'} \tilde{g}(b_{k'}, s_{l'})\hat{g}(b_{k'}, s_{l'})}$$

must hold in an equilibrium.

Since

$$\mathcal{U}_b(b_k) + \mathcal{U}_s(s_l) \leq b_k - s_l,$$

$\exists p_{b_k}, q_{s_l} \geq 0$ such that

$$\mathcal{U}_b(b_k) = b_k - p_{b_k} \quad \text{and} \quad \mathcal{U}_s(s_l) = q_{s_l} - s_l.$$

Given $\mathcal{M}(g)$, let $\mathcal{U}(\mathcal{M}(g))$ be the set of the interim expected payoff vectors induced by an incentive compatible mechanism in $\mathcal{M}(g)$. Let $\tilde{\nu}$ be a probability distribution over the set of all incentive compatible trading mechanisms $\mathcal{U}(\mathcal{M}(g))$. Let ν be the probability distribution over D, induced by $\tilde{\nu}$. We can regard ν_{kl} as the marginal distribution of ν over the space of (p, q) pair drawn for b_k buyer and s_l seller.

5. Concluding Remarks

5.1. **Exogenous exit probability.** One of the assumptions often made in the existing models (e.g., Satterthwaite and Shneyerov (2008)) is to limit the total number of searches by an agent by forcing the agent to exit the market after a certain number of failed attempts to form a long term relationship. Those who cannot trade can leave the market without changing our limit result at all. This is due to our matching technology according to which the presence of those who never trade may only decrease the probability of matching with those who trade with a positive probability. Thus, this effect is washed away if we make the matching more frequent by letting Δ become small.

5.2. **Rate of convergence.** Even though the equilibrium outcome of the economy converges to the competitive outcome as friction vanishes, it is important to see how fast the convergence is. It is beyond the scope of this paper to characterize the rate of the convergence to the competitive equilibrium. Yet, a numerical example indicates the convergence rate appears to be very slow.

The numbers in Table 1 are generated under the assumption that

$$b_1 = 10,\ b_2 = 5;\ x_1 = 0.4,\ x_2 = 0.6;\ s_1 = 2,\ s_2 = 8;\ y_1 = 0.6,\ y_2 = 0.4$$

where $u_{bk}(p) = b_k - p$ and $u_{sl}(q) = q - s_l$. Recall that $\beta = e^{-\Delta b}$ and $\delta = e^{-\Delta d}$. We assume that $b = 1$ and $d = 10$. In this case, the competitive equilibrium price is 5, and 60% of each party trade. Thus, $W_b^1 \to 5$ and $W_s^1 \to 3$, while $W_b^2, W_s^2 \to 0$. At the same time, $z_b = z_b^1 + s_b^2$ and $z_s = z_s^1 + z_s^2$ converge to 0.4. Note that since $x_2 = 0.6$, 1/3 of b_2 buyers trade at a price close to $b_2 = 5$, while the remaining 2/3 of b_2 buyers cannot trade.

Δ	0.1	0.1^5	0.1^9
z_b^1	0.2414	0.0176	0.0007
z_b^2	0.4531	0.4033	0.4002
z_s^1	0.3423	0.0210	0.0011
z_s^2	0.3522	0.4000	0.4000
W_b^1	0.9733	3.4312	4.6344
W_b^2	0.1778	0.0057	0.0050
W_s^1	0.8881	2.9005	2.9915
W_s^2	0.0336	0.0049	0.0050
π_{11}	0.5888	0.0435	0.0022
π_{12}	0.2465	0	0
π_{21}	0.4156	0.0010	7.0041×10^{-6}
π_{22}	0	0	0

TABLE 1. Since $b_2 < s_2$, $\pi_{22} = 0\ \forall \Delta > 0$. As $\Delta \to 0$, $\pi_{12}, \pi_{21} \to 0$, even though $b_1 > s_2$ and $b_2 > s_1$. The rate of convergence of W_b^1 to 5 is particularly slow. Because of the rounding error, $z_b^1 + z_b^2 \neq z_s^1 + z_s^2$ may occur by a small amount. We treat a positive number as zero, if it is smaller than 0.1×10^{-16}.

Appendix

Throughout the appendix, we write the equilibrium thresholds of b_k buyer and q_l seller as p_k and q_l, defined as (3.18) and (3.19) respectively.

Appendix A. Proof of Theorem 2.4

Define a mixed reporting strategy of b_k buyer as

$$\varphi_b^k \in \Delta^K$$

as a probability distribution over K types. Similarly, we define a reporting strategy of s_l seller as $\varphi_s^l \in \Delta^L$. Let $\varphi_b = (\varphi_b^k)_k$ and $\varphi_s = (\varphi_s^l)$. By k and l (without prime), we mean the true types of b_k buyer and s_l seller. By k' and l', we mean the reported types of b_k buyer and s_l seller.

Given a profile of reporting strategies, and the profile of expected utility (W_b, W_s), each player uses the "equilibrium" threshold to decided whether or not accept the proposal. That is, b_k buyer accepts p if

$$u_{bk}(p) \geq W_b^k$$

and similarly, s_l seller accepts q if

$$u_{sl}(q) \geq W_s^l$$

so that we can focus on the existence of an equilibrium reporting strategy, when $\nu_{k'l'}$ is determined by a pair of reported types rather than a pair of true types.

Define $(\tilde{z}_b, \tilde{z}_s)$ as the profile of the inverses of the components of (z_b, z_s):

$$\tilde{z}_b^k = \frac{1}{z_b^k} \text{ and } \tilde{z}_s^l = \frac{1}{z_s^l}.$$

Fix $(W_b, W_s; \tilde{z}_b, \tilde{z}_s; \varphi_b, \varphi_s)$ which is an element in a compact convex subset of $\mathbb{R}^{K^2+K+L^2+L}$. We can recover (z_b, z_s) from $(\tilde{z}_b, \tilde{z}_s)$, and then, compute (μ_b, μ_s) from (z_b, z_s) through a continuous function. Define

$$\Pi_{kk'll'} = \{(p,q) | \ u_{bk}(p) \geq W_b^k, \ u_{sl}(q) \geq W_s^l, \ p \geq q\},$$

and

$$\pi(k, k'; l, l') = \nu_{k'l'}(\Pi_{kk'll'})$$

as the probability of reach agreement when b_k buyer reports $b_{k'}$ and s_l seller reports $s_{l'}$, and then both players use the true thresholds.

Given φ_s, the expected payoff of b_k buyer if he reports $b_{k'}$ is

$$\overline{W}_b(k, k', \varphi_s) = \frac{\beta \sum_{l'} \sum_l \pi(k, k'; l, l') \varphi_s(l'|l) \mu_s^l \mathsf{E}(u_{bk}(p) \mid \Pi_{kl})}{1 - \beta\delta + \beta \sum_{l'} \sum_l \pi(k, k'; l, l') \varphi_s(l'|l) \mu_s^l}.$$

Then, we can define

$$\overline{W}_b^k = \max_{k'} \overline{W}_b(k, k', \varphi_s)$$

and $\overline{\Phi}_b(\cdot|k)$ as the set of probability distributions over

$$\arg\max_{k'} \overline{W}_b(k, k', \varphi_s)$$

and $\overline{\varphi}_b(\cdot|k)$ as a generic element of $\overline{\Phi}_b(\cdot|k)$ which is convex and compact. Similarly, we can define $\overline{W}_s^l$, $\overline{\Phi}_s^l$ and $\overline{\varphi}_s^l$. Also, we can compute $\overline{z}_b$ and $\overline{z}_s$ using $\overline{W}_s, \overline{W}_b, \overline{\varphi}_b, \overline{\varphi}_s$ in conjunction with the balance equation, which should be modified to incorporate a randomized reporting strategy. Define

$$\overline{\pi}_{k\tilde{l}} = \sum_{l'} \pi(k, k'; \tilde{l}, l') \overline{\varphi}_s(l'|\tilde{l}) \overline{\varphi}_b(k'|k)$$

as the probability of reaching agreement if $s_{\tilde{l}}$ seller reports according to $\varphi_s(\cdot|\tilde{l})$ and b_k buyer reports according to $\varphi_b(\cdot|k)$. Then, we can define

$$\frac{1}{\overline{\tilde{z}}_b^k} = \overline{z}_b^k = \frac{x_k(1-\delta)}{1 - \delta + \sum_{\tilde{l}} \mu_s^{\tilde{l}} \overline{\pi}_{k\tilde{l}}}$$

for given μ_s. Similarly, we define $\overline{\tilde{z}}_s^l$ and $\overline{z}_s^l$ for given μ_b.

Note that $(\overline{\tilde{z}}_b, \overline{\tilde{z}}_s)$ is a linear function of $(\overline{\varphi}_s, \overline{\varphi}_b)$, whose domain $(\overline{\Phi}_b, \overline{\Phi}_s)$ is compact and convex. Thus, the set $\overline{\tilde{Z}}$ of $(\overline{\tilde{z}}_b, \overline{\tilde{z}}_s)$'s is a compact convex subset of $\mathbb{R}_+^{K+L}$.[15]

Consider a mapping

$$(W_b, W_s; \tilde{z}_b, \tilde{z}_s; \varphi_b, \varphi_s) \mapsto (\overline{W}_b, \overline{W}_s; \overline{\tilde{Z}}; \overline{\Phi}_b, \overline{\Phi}_s)$$

which is a compact convex valued correspondence with a close graph in a compact convex subset of an Euclidean space. Thus, by Kakutani's fixed point theorem, it has a fixed point, from which the existence of an undominated stationary equilibrium follows.

[15]We update μ_b and μ_s after we compute $\overline{z}_b$ and $\overline{z}_s$ from $\overline{Y}$.

Appendix B. Proof of Lemma 3.1

We show the first line only. The second line follows the same logic. Observe first

$$\pi_{kl}\mathsf{E}[u_{bk}(p) - W_b^k|\Pi_{kl}] = \int\int_{\Pi_{kl}} (u_{bk}(p) - W_b^k)\nu_{kl}(p,q)dpdq.$$

We have

$$\begin{aligned} \inf_{p,q,z} \nu_{kl}^z(p,q) \int\int_{\Pi_{kl}} (u_{bk}(p) - W_b^k)dpdq &\leq \int\int_{\Pi_{kl}} (u_{bk}(p) - W_b^k)\nu_{kl}^z(p,q)dpdq \\ &\leq \sup_{p,q,z} \nu_{kl}^z(p,q) \int\int_{\Pi_{kl}} (u_{bk}(p) - W_b^k)dpdq. \end{aligned}$$

Letting $\underline{A}_b$ sufficiently small and $\overline{A}_b$ sufficiently large, and substituting these expressions into (2.14), we obtain the inequality as desired.

Appendix C. Proof of Lemma 3.2

We show the monotonicity of the threshold prices of buyers. Consider

$$\rho_{bk}(p) = u_{bk}(p) - W_b^k = u_{bk}(p) - (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} \sum_{j=1}^{K} \pi_{kj}\mathsf{E}(u_{bk}(p') \mid \Pi_{kj}).$$

With (2.2), (2.3) and (2.4),

$$\frac{d\rho_{bk}(p)}{dp} < 0$$

and

$$\rho_{b,k+1}(p) < \rho_{bk}(p) \qquad \forall p \geq 0.$$

The conclusion follows from the observation that the equilibrium threshold p_k is defined implicitly as

$$\rho_{bk}(p_k) = 0.$$

Appendix D. Proof of Lemma 3.3

Recall

$$\Pi_{kl} = \{(p,q)|p_k \geq p, \quad q \geq q_l\}$$

and $\pi_{kl} = \nu_{kl}(\Pi_{kl})$, where p_k is the equilibrium threshold price of b_k buyer and q_l is the equilibrium threshold price of s_l seller. By the monotonicity,

$$\Pi_{kl} \supset \Pi_{k+1,l}$$

holds. Since every ν_{kl} is bounded away from zero on D, $\nu_{kl}(\Pi_{kl}) = 0$ implies $\nu_{kl}(\Pi_{k+1,l}) = 0$. The same logic applies to $\Pi_{k,l+1}$.

Appendix E. Proof of Lemma 3.4

Suppose otherwise. Then, we can find a sequence of (Δ_n) converging to zero such that for each Δ_n, there exists an equilibrium where

$$\pi_{k^*l} > 0 \qquad \text{and} \qquad \pi_{kl^*} > 0 \tag{E.21}$$

hold for some k and l, and all (finitely many) relevant variables are convergent. Take such k and l. This implies that type k^* buyers and type l sellers trade, and so do type l^* sellers and type k buyers. Then we must have

$$b_k > s_{l^*} > b_{k^*} > s_l.$$

Since type k^* buyers and type l sellers trade with a positive probability, we have

$$p_{k^*} > q_l$$

for each Δ_n. Similarly,

$$p_k > q_{l^*}$$

holds for each Δ_n. By individual rationality, $b_k \geq p_k$, $b_{k^*} \geq p_{k^*}$, $q_{l^*} \geq s_{l^*}$, and $q_l \geq s_l$ hold. Since the equilibrium threshold price is monotonic with respect to the type of the agent,

$$p_k \geq p_{k^*} \quad \text{and} \quad q_{l^*} \geq q_l.$$

Combining these inequalities, we obtain

$$p_k \geq s_{l^*} > b_{k^*} \geq q_l$$

for each Δ_n. Noticing that s_{l^*} and b_{k^*} are exogenous parameters, we take the limit and obtain

$$\lim_{\Delta_n \to 0} p_k - q_l \geq \lim_{\Delta_n \to 0} s_{l^*} - b_{k^*} > 0,$$

which implies

$$\lim_{\Delta_n \to 0} \pi_{kl} > 0.$$

Then, since

$$\lim_{\Delta_n \to 0} W_s^l \leq \lim_{\Delta_n \to 0} \frac{\beta \overline{A}_s}{1 - \beta\delta} \sum_{k'} \mu_b^{k'} (\pi_{k'l})^{1+\alpha} < \infty$$

holds, we have

$$\lim_{\Delta_n \to 0} \frac{\mu_b^k}{1 - \beta\delta} < \infty. \tag{E.22}$$

Note

$$\mu_b^k = \frac{z_b^k}{\sum_{k'} z_b^{k'}}$$

and

$$\lim_{\Delta_n \to 0} \frac{\mu_b^k}{1 - \beta\delta} = \frac{\frac{z_b^k}{1-\beta\delta}}{\sum_{k'} z_b^{k'}} < \infty.$$

Since

$$\sum_{k'} z_b^{k'} \leq X_K,$$

we have

$$\lim_{\Delta_n \to 0} \frac{z_b^k}{1 - \beta\delta} < \infty. \tag{E.23}$$

One can write (2.11) as

$$z_b^k = \frac{1 - \delta}{\sum_l \mu_s^l \pi_{kl} + 1 - \delta} x_b^k.$$

From (E.23), we know

$$\lim_{\Delta_n \to 0} \frac{z_b^k}{1 - \delta} = \lim_{\Delta_n \to 0} \frac{x_b^k}{\sum_l \mu_s^l \pi_{kl} + 1 - \delta} < \infty. \tag{E.24}$$

Since $x_b^k > 0$ is a constant parameter, $\exists l'$ such that

$$\lim_{\Delta_n \to 0} \mu_s^{l'} \pi_{kl'} > 0.$$

Otherwise, the left hand side of (E.24) diverges to infinity. Take such an l'. The boundedness of $\mu_s^{l'}$ and $\pi_{kl'}$ implies

$$\lim_{\Delta_n \to 0} \mu_s^{l'} > 0, \quad \text{and} \quad \lim_{\Delta_n \to 0} \pi_{kl'} > 0$$

Then, (3.16) together with the non-negativeness of each term therein implies that

$$W_b^k \geq \frac{\beta \underline{A}_b}{1-\beta\delta} \mu_s^{l'} (\pi_{kl'})^{1+\alpha} \to \infty$$

as Δ_n goes to zero. This is impossible because W_b^k must be uniformly bounded.

Appendix F. Proof of Lemma 3.5

Suppose otherwise. Then, Lemma 3.4 implies that there exists a sequence (Δ_n) converging to zero such that for each Δ_n, there exists an equilibrium in which

$$\forall k (\pi_{kl^*} = 0).$$

holds, and all (finitely many) relevant variables are convergent. This implies that $W_s^{l^*} = 0$ holds. Moreover, $\pi_{kl^*} = 0$ implies

$$0 < p_k - q_{l^*} \to 0$$

as $\Delta \to 0$. Recall that $u_{sl^*}(s_{l^*}) = 0$ and $W_s^{l^*} = 0$. By the definition,

$$u_{bk}^{-1}(W_b^k) - u_{sl^*}^{-1}(W_s^{l^*}) \to 0 = u_{bk}^{-1}(W_b^k) - u_{sl^*}^{-1}(0) = u_{bk}^{-1}(W_b^k) - s_{l^*},$$

where u_{bk}^{-1} and $u_{sl^*}^{-1}$ are the inverse functions of u_{bk} and u_{sl^*}, respectively. Note that $u_{bk}(p)$ is strictly decreasing function of p, and that $\forall k = 1, \ldots, k^*-1$, $b_k > s_{l^*}$. Thus,

$$0 = u_{bk}(b_k) < u_{bk}(s_{l^*}) = W_b^k \qquad \forall k = 1, \ldots, k^*-1.$$

Hence,

$$W_b^k > 0 \tag{F.25}$$

holds for all $k = 1, \ldots, k^*-1$ independently of $\Delta_n > 0$.

By Lemma 3.3, $\forall l \geq l^*$, $\forall k$, $\pi_{kl} = 0$. Thus, $\forall l \geq l^*$, $z_s^l = y_l$. Since $Y_{l^*-1} < X_{k^*-1}$,

$$\sum_{l'=1}^{L} z_s^{l'} \geq \sum_{l=l^*}^{L} y_{l'} = 1 - Y_{l^*-1} > 1 - X_{k^*-1}.$$

Note that the strict inequality is independent of $\Delta_n > 0$. Since the equal mass of unmatched buyers and unmatched sellers must exist, $\exists k < k^*$ such that

$$\lim_{\Delta_n \to 0} z_b^k > 0.$$

Take such a k. From (2.10), we have

$$z_b^k = \frac{1-\delta}{\sum_{l=1}^{l^*-1} \mu_s^l \pi_{kl} + 1 - \delta} x_b^k,$$

since $\pi_{kl} = 0$ holds for all $l \geq l^*$. In order for z_b^k to be uniformly bounded away from 0,

$$\lim_{\Delta_n \to 0} \frac{\mu_s^l \pi_{kl}}{1-\delta} < \infty \qquad \forall l.$$

Since $z_b^k > 0$ holds uniformly of $\Delta_n > 0$, $\mu_b^k > 0$ holds uniformly of $\Delta_n > 0$. Then, (3.17) implies that

$$\lim_{\Delta_n \to 0} \frac{\pi_{kl}^{1+\alpha}}{1-\beta\delta} < \infty$$

which implies that

$$\lim_{\Delta_n \to 0} \pi_{kl} = 0.$$

Thus,

$$\lim_{\Delta_n \to 0} \frac{\mu_s^l \pi_{kl}^{1+\alpha}}{1-\delta} = 0$$

holds for all l. Then, we have

$$W_b^k \leq \frac{\beta \overline{A}_b}{1-\beta\delta} \sum_{l=1}^{l^*-1} \mu_s^l (\pi_{kl})^{1+\alpha} \to 0$$

as $\Delta_n \to 0$. This contradicts to (F.25).

Appendix G. Proof of Lemma 3.6

Suppose the contrary, i.e., that there exist $l < l^*$ and a sequence (Δ_n) converging to zero such that

$$\lim_{\Delta_n \to 0} z_s^l > 0$$

holds, and therefore,

$$\lim_{\Delta_n \to 0} \mu_s^l > 0,$$

and all other (finitely many) variables are convergent. Take such an l. Then from (3.16), we have

$$\lim_{\Delta_n \to 0} \frac{(\pi_{kl})^{1+\alpha}}{1-\beta\delta} < \infty \tag{G.26}$$

for all k.

We claim that

$$\lim_{\Delta_n \to 0} W_s^l > 0. \tag{G.27}$$

Suppose the contrary, i.e., that

$$\lim_{\Delta_n \to 0} W_s^l = 0.$$

By Lemma 3.5, $\pi_{kl^*} > 0$ for some k along the sequence (if not, find a subsequence with this property and continue the rest of the proof). Take such a k. Then, we have

$$p_k > q_{l^*}.$$

Since the threshold prices are monotonic with respect to the type of the seller,

$$q_{l^*} \geq q_l = u_{sl}^{-1}(W_s^l) \to u_{sl}^{-1}(0) = s_l$$

under the hypothesis of the proof. In an equilibrium, $q_{l^*} \geq s_{l^*}$ by the individual rationality. Thus,

$$p_k > q_{l^*} \geq s_{l^*} > s_l = \lim_{\Delta_n \to 0} q_l.$$

Note that $s_{l^*} - s_l$ is independent of Δ_n. Thus,

$$\lim_{\Delta_n \to 0} p_k - q_l > 0$$

which implies

$$\lim_{\Delta_n \to 0} \pi_{kl} > 0.$$

Then,

$$W_b^k \geq \frac{\beta \underline{A}_b \sum_{l'} \mu_s^{l'} (\pi_{kl'})^{1+\alpha}}{1-\beta\delta} \geq \frac{\beta \underline{A}_b \mu_s^l (\pi_{kl})^{1+\alpha}}{1-\beta\delta} \to \infty,$$

which is impossible since W_b^k is uniformly bounded over $\Delta > 0$. This contradiction proves (G.27).

Since $\pi_{kl} = 0$ holds for all $k \geq k^*$ by Lemma 3.5, (G.27) implies

$$0 < \lim_{\Delta_n \to 0} W_s^l \leq \lim_{\Delta_n \to 0} \frac{\beta \overline{A}_s \sum_{k'=1}^{k^*-1} \mu_b^{k'} (\pi_{k'l})^{1+\alpha}}{1-\beta\delta} < \infty. \tag{G.28}$$

The last inequality implies that for all $k' = 1, \dots, k^* - 1$, there exists $B_k \geq 0$ such that

$$\mu_b^{k'} (\pi_{k'l})^{1+\alpha} = B_{k'} \Delta + o(\Delta) \tag{G.29}$$

where

$$\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta}=0.$$

Since $\lim_{n\to\infty} z_s^l > 0$, (2.11) implies

$$0<\lim_{\Delta_n\to 0}\frac{\sum_{k'}\mu_b^{k'}\pi_{k'l}}{1-\delta}<\infty. \tag{G.30}$$

The right inequality implies that $\forall k' \in \{1,\ldots,k^*-1\}$, $\exists B'_{k'}\geq 0$

$$\mu_b^{k'}\pi_{k'l}=B'_{k'}\Delta+o(\Delta) \tag{G.31}$$

The left inequality of (G.30) implies that $\exists \tilde{k}\in\{1,\ldots,k^*-1\}$ such that $B'_{\tilde{k}}>0$.

We claim that $B_{\tilde{k}}>0$. A simple calculation shows that

$$\frac{\mu_b^{\tilde{k}}\pi_{\tilde{k}l}^{1+\alpha}}{\Delta}=\left[B'_{\tilde{k}}\left(\frac{\mu_b^{\tilde{k}}}{\Delta}\right)^{\alpha}+o(\Delta)\left(\frac{\mu_b^{\tilde{k}}}{\Delta}\right)^{\alpha}\right]^{1+\alpha}.$$

Since

$$\lim_{\Delta\to 0}\frac{\mu_b^{k'}}{\Delta}>0,$$

the right hand side is bounded away from 0 as $\Delta\to 0$. Thus, $B_{\tilde{k}}>0$.

Note

$$\frac{\mu_b^{\tilde{k}}\pi_{\tilde{k}l}^{1+\alpha}}{\mu_b^{\tilde{k}}\pi_{\tilde{k}l}}=\pi_{\tilde{k}l}^{\alpha}=\frac{B_{\tilde{k}}\Delta+o(\Delta)}{B'_{\tilde{k}}\Delta+o(\Delta)}=\frac{B_{\tilde{k}}}{B'_{\tilde{k}}}+o(\Delta).$$

Then,

$$\lim_{\Delta_n\to 0}\frac{\pi_{\tilde{k}l}^{1+\alpha}}{1-\delta}=\infty \tag{G.32}$$

which contradicts to (G.26).

Appendix H. Proof of Lemma 3.7

Note that Lemma 3.5 implies $z_b^k = x_b^k > 0$ for all $k\geq k^*$. Lemma 3.6 implies that $z_s^l\to 0$. Recall $X_{k^*-1}<Y_{l^*}$. In order to maintain the same mass of buyers and sellers in the pool,

$$\liminf_{\Delta\to 0} z_s^l>0$$

for $l\geq l^*$. Therefore,

$$\lim_{\Delta\to 0}\mu_s^l=0$$

holds for all $l<l^*$.

Appendix I. Proof of Lemma 3.8

By Lemma 3.5, $\exists k<k^*$ $\pi_{kl^*}>0$. We know $\forall\Delta>0$, $\mu_b^k>0$. Thus,

$$W_s^{l^*}=\frac{\beta A_s\sum_{k'}\mu_b^{k'}\pi_{k'l^*}}{1-\beta\delta}\geq\frac{\beta A_s\mu_b^k\pi_{kl^*}}{1-\beta\delta}>0.$$

To prove

$$\lim_{\Delta\to 0}W_s^{l^*}=0$$

by way of contradiction, suppose the contrary: there exists a sequence (Δ_n) converging to zero such that

$$\lim_{\Delta_n\to 0}W_s^{l^*}>0$$

holds, and all other (finitely many) variables are convergent. From (3.16), this implies that

$$\lim_{\Delta_n \to 0} \frac{\mu_b^k \pi_{kl^*}^{1+\alpha}}{1-\beta\delta} > 0 \tag{I.33}$$

holds for some k. Take such a k. Lemmata 3.3, 3.5, and 3.6 together with the one-to-one matching rule give us $\lim_{\Delta_n \to 0} z_s^{l^*} > 0$. Then, from (2.11) and Lemma 3.7, we have

$$0 < \lim_{\Delta_n \to 0} \frac{\sum_{k'} \mu_b^{k'} \pi_{k'l^*}}{1-\delta} = \lim_{\Delta_n \to 0} \frac{y_{l^*} - z_s^{l^*}}{z_s^{l^*}} < \infty.$$

Lemma 3.6 implies $\lim_{\Delta_n \to 0} \mu_s^{l^*} > 0$. Therefore, (2.10) implies

$$\lim_{\Delta_n \to 0} \frac{\mu_s^{l^*} \pi_{kl^*}}{1-\delta} < \infty$$

for all k. Due to $\lim_{\Delta_n \to 0} \mu_s^{l^*} > 0$, the above expression implies

$$\lim_{\Delta_n \to 0} \frac{\pi_{kl^*}}{1-\delta} < \infty,$$

which implies

$$\lim_{\Delta_n \to 0} \frac{\pi_{kl^*}^{1+\alpha}}{1-\delta} = 0,$$

a fortiori,

$$\lim_{\Delta_n \to 0} \frac{\mu_b^k \pi_{kl^*}^{1+\alpha}}{1-\beta\delta} = 0$$

for all k, which is a contradiction to (I.33).

Appendix J. Proof of Lemma 3.9

From (3.16), we have

$$\beta \underline{A}_b \frac{\mu_s^{l^*} \pi_{kl^*}^{1+\alpha}}{1-\beta\delta} \le W_b^k < \infty.$$

Since $\lim_{\Delta \to 0} \mu_s^{l^*} > 0$ holds due to Lemma 3.7, we have

$$\lim_{\Delta \to 0} \frac{\pi_{kl^*}^{1+\alpha}}{1-\beta\delta} < \infty.$$

Thus, $\lim_{\Delta \to 0} \pi_{kl^*} = 0$ must hold.

Appendix K. Proof of Lemma 3.10

Due to Lemma 3.5, it suffices to show the statement for $k < k^*$. Lemma 3.9 implies that $\forall k < k^*$,

$$\limsup_{\Delta \to 0} p_k - q_{l^*} \le 0.$$

By Lemma 3.8,

$$\lim_{\Delta \to 0} W_s^{l^*} = 0$$

which implies that

$$q_{l^*} \to s_{l^*}.$$

Since $q_l \ge s_l > s_{l^*}$, and $s_l - s_{l^*}$ is independent of Δ,

$$\liminf_{\Delta \to 0} q_l - q_{l^*} > 0.$$

Since $\forall l > l^*$,

$$0 \le W_s^l \le W_s^{l^*} \to 0,$$

$$\lim_{\Delta \to 0} q_l - s_l = 0.$$

Thus, $\exists\overline{\Delta} > 0$ such that $\forall \Delta \in (0, \overline{\Delta})$,

$$p_k - q_l < 0 \qquad \forall k < k^*, \forall l > l^*$$

from which the conclusion follows.

Appendix L. Proof of Lemma 3.11

Since $\mu_s^{l^*} > 0$ due to Lemma 3.7 and $\lim_{\Delta\to 0} W_s^{l^*} = 0$ due to Lemma 3.8, type $k < k^*$ buyer can obtain a positive payoff bounded away from zero with a positive conditional probability $\pi'_{kl^*} > 0$ by setting its threshold at, say, $\frac{b_{k^*-1}+s_{l^*}}{2}$. Therefore, 2.12 together with $\mu_s^{l^*} > 0$ due to Lemma 3.7 implies

$$W_b^k \geq \frac{\beta\mu_s^{l^*}\pi'_{kl^*}\mathsf{E}\left(b_k - \frac{b_{k^*-1}+s_{l^*}}{2}|\Pi'_{kl^*}\right)}{1-\beta\delta+\sum_{l'\neq l^*}^{L}\mu_s^{l'}\pi_{kl'}+\mu_s^{l^*}\pi'_{kl^*}} > 0.$$

Appendix M. Proof of Lemma 3.12

Suppose the contrary, i.e., that there exists a sequence (Δ_n) converging to zero such that

$$\lim_{\Delta_n\to 0} z_b^k > 0$$

holds for some $k < k^*$, and all other (finitely many) variables are convergent, so that we have

$$\lim_{\Delta_n\to 0} \mu_b^k > 0$$

as a direct implication. Take such an k. From Lemma 3.7, we have $z_s^{l^*} > 0$. From (2.11) and $\mu_b^k > 0$, it is necessary that

$$\lim_{\Delta_n\to 0} \frac{\pi_{kl^*}}{1-\delta} < \infty.$$

Then we have

$$\lim_{\Delta_n\to 0} \frac{\pi_{kl^*}^{1+\alpha}}{1-\beta\delta} = 0. \tag{M.34}$$

On the other hand,

$$\lim_{\Delta_n\to 0} \frac{\beta\underline{A}_s\mu_b^k\pi_{kl}^{1+\alpha}}{1-\beta\delta} \leq \lim_{\Delta_n\to 0} W_s^l < \infty$$

implies that

$$\lim_{\Delta_n\to 0} \frac{\pi_{kl}^{1+\alpha}}{1-\beta\delta} < \infty \tag{M.35}$$

holds for $l < l^*$ since we have $\lim_{\Delta_n\to 0}\mu_b^k > 0$ under the hypothesis of the proof. Note that (3.16) together with Lemma 3.10 implies

$$W_b^k \leq \frac{\beta\overline{A}_b\left[\sum_{l'=1}^{l^*-1}\mu_s^{l'}\pi_{kl'}^{1+\alpha}+\mu_s^{l^*}\pi_{kl^*}^{1+\alpha}\right]}{1-\beta\delta}$$

for a sufficiently small $\Delta > 0$. By Lemma 3.7 $\lim_{\Delta_n\to 0}\mu_s^l = 0$, which, together with (M.35) implies the first term in the bracket must vanish, and (M.34) implies that the second term must vanish, as $\Delta_n \to 0$. Thus, $\lim_{\Delta_n\to 0} W_b^k = 0$, which is a contradiction to Lemma 3.11.

Appendix N. Proof of Lemma 3.13

Suppose the contrary: $\exists k < k^*$ and $\overline{\Delta} > 0$ such that $\pi_{kl^*} = 0 \; \forall \Delta \in (0, \overline{\Delta})$, or equivalently

$$p_k \leq q_{l^*} \qquad \forall \Delta < \overline{\Delta}.$$

By Lemma 3.2, p_k is an increasing function of b_k,

$$p_{k^*} \leq q_{l^*}$$

We have already established

$$\lim_{\Delta \to 0} W_b^{k^*-1} > 0.$$

Thus, $\exists l < l^*$, $\forall \Delta \in (0, \overline{\Delta})$

$$p_{k^*-1} - q_l > 0 \tag{N.36}$$

or equivalently,

$$\pi_{k^*-1,l} > 0.$$

On the other hand, Lemma 3.4 implies that $\exists k < k^* - 1$ such that

$$\lim_{\Delta \to 0} p_k - q_{l^*} = 0. \tag{N.37}$$

Thus, (N.36) and (N.37) imply $\exists k, l$ such that

$$\lim_{\Delta \to 0} p_k - q_l > 0$$

or equivalently,

$$\lim_{\Delta \to 0} \pi_{kl} > 0.$$

Recall that $\exists \alpha > 0$ such that

$$W_s^l \leq \frac{\beta \overline{A}_s \sum \mu_b^{k'} \pi_{k'l}^{1+\alpha}}{1 - \beta\delta} < \infty$$

holds uniformly over $\Delta > 0$. Thus, for k and l identified by the claim,

$$\lim_{\Delta \to 0} \frac{\mu_b^k \pi_{kl}^{1+\alpha}}{1 - \beta\delta} < \infty.$$

Since $\lim_{\Delta \to 0} \pi_{kl} > 0$,

$$\lim_{\Delta \to 0} \frac{\mu_b^k}{1 - \beta\delta} < \infty.$$

Since $\sum_{k'=1}^{K} z_{k'} \leq X_K$,

$$\lim_{\Delta \to 0} \frac{z_b^k}{1 - \beta\delta} < \infty.$$

By (2.10),

$$\frac{z_b^k}{1 - \delta} = \frac{x_b^k}{\sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'} + 1 - \delta}.$$

Since $x_b^k > 0$ is a constant,

$$\lim_{\Delta \to 0} \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'} > 0.$$

Since $\mu_s^{l'}$ is uniformly bounded,

$$\lim_{\Delta \to 0} \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'}^{1+\alpha} > 0,$$

Hence,

$$W_b^k \geq \frac{\beta \underline{A}_b \sum_{l'=1}^{L} \mu_s^{l'} \pi_{kl'}^{1+\alpha}}{1-\beta\delta} \to \infty$$

as $\Delta \to 0$, which is impossible, because W_b^k is uniformly bounded over $\Delta > 0$.

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