Search, Adverse Selection, and Market Clearing

By

In-Koo Cho and Akihiko Matsui

April 2018

CENTER FOR RESEARCH AND EDUCATION FOR POLICY EVALUATION
DISCUSSION PAPER NO. 16

CENTER FOR RESEARCH AND EDUCATION FOR POLICY EVALUATION (CREPE)
THE UNIVERSITY OF TOKYO
http://www.crepe.e.u-tokyo.ac.jp/

SEARCH, ADVERSE SELECTION, AND MARKET CLEARING

IN-KOO CHO AND AKIHIKO MATSUI

Abstract. This paper investigates the conditions under which adverse selection causes the coexistence of involuntary unemployment and involuntary vacancy even if search friction vanishes in a dynamic decentralized trading model. An economy is populated by high quality and low quality sellers as well as buyers. In each period, sellers who know the quality of the good and buyers who do not observe the quality are randomly matched in pairs. For each pair, a price is randomly drawn. If either party disagrees, then the two agents return to the pool. If both parties agree, then the trade occurs, and the two agents leave the pool, generating surplus from trading. The long term agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, the seller and the buyer return to the matching market. We demonstrate that involuntary unemployment and involuntary vacancy coexist under adverse selection if and only if agents are sufficiently patient relative to the rate of dissolution when there are more sellers than buyers, and that they coexist whenever there are no less buyers than sellers. We also show that the main results of the baseline model are carried over to models with different bargaining protocols such as a take-it-or-leave-it offer game.

Keywords: Involuntary unemployment, Involuntary vacancy, Matching, Search friction, Adverse selection, Undominated equilibrium, Market clearing

1. Introduction

Persistent coexistence of involuntary unemployment and involuntary vacancy in labor markets is a major challenge to general equilibrium theory.[1] Search theoretic models have been developed to explain this coexistence as an equilibrium outcome

Date: September 4, 2017.

We are grateful for helpful conversation with Asher Wolinsky, Briana Chang and John Moore, and helpful comments by three anonymous referees and the editor. Financial support from the National Science Foundation, the Japan Society for the Promotion of Science and Korea Research Foundation (2015S1A5A2A01011999) is gratefully acknowledged. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation, Korea Research Foundation, Federal Reserve Bank or the Japan Society for the Promotion of Science.

[1]Our notions of involuntary unemployment and involuntary vacancy are inspired by Keynes (1936), who differentiates involuntary unemployment from *frictional* unemployment which arises as a consequence of (search) friction. To be considered "involuntary" unemployment in our paper, for example, an agent must remain unemployed, even in the limit as search friction vanishes.

in an economy with non-negligible amount of friction.[2] The goal of the present paper is to investigate whether or not and how adverse selection can be an alternative source, other than search friction, that prevents the market from clearing itself. We demonstrate that the market fails to clear under adverse selection – even in the limit of search friction vanishing – if and only if agents are sufficiently patient.

We say that *involuntary unemployment* occurs if the number of people who are willing to work at prevailing prices is greater than the number of people who are actually working, even in the limit as search friction vanishes. Similarly, we say that *involuntary vacancy* occurs if the number of people whom some other people are willing to hire at prevailing prices is greater than the number of people who are actually hired, even as search friction vanishes. While the focus of this paper is not a labor market *per se*, we choose to use these terms partly because the failure of market clearing is extensively studied in the context of the labor market.

Our model is built on a dynamic decentralized trading model (e.g., Rubinstein and Wolinsky (1985) and Cho and Matsui (2017)) with four important features. First, a seller has private information about the common value component of the product, such as the quality (Chang (2012)). Second, the matching technology is efficient in the sense that an agent in the short side of the pool is matched with probability 1, while the long side is rationed (Rubinstein and Wolinsky (1985)). Third, in order to consummate the match, the seller and the buyer must agree upon a price at which the good is delivered. We model the bargaining process as a random proposal model (Burdett and Wright (1998)). We choose this bargaining protocol mainly for analytical convenience. The main results of this model are carried over to other models with different bargaining protocols such as a take-it-or-leave-it offer game. Fourth, we shut down free entry of agents in the baseline model, in order to fix the mass of sellers and buyers in the economy finite. This is an important feature to define the *rates* of unemployment and vacancy precisely. After completely analyzing the baseline model, we extend the analysis to a class of models in which buyers freely enter the economy, while paying the vacancy cost to stay in the market (Mortensen and Pissarides (1994)).

An economy is populated by two unit mass of two types of infinitesimal (infinitely-lived) sellers: one unit mass of high quality sellers, and the other unit mass of low quality sellers. There is a finite mass of infinitesimal buyers. In each period, sellers who know the quality of the good and buyers who do not observe the quality are randomly matched in pairs with a long side being rationed. For each pair, a price is randomly drawn from a continuous density function bounded away from zero in a certain range. If either party disagrees, then the two agents return to the pool, waiting for the next chance to be matched to another agent. If both parties agree,

[2]The source of friction can be the time elapsed between two matches for each agent or the inefficiency of matching technology (e.g., Mortensen and Pissarides (1994), Burdett, Shi, and Wright (2001) and Lagos (2000)).

then the trade occurs, and the two agents leave the pool of unmatched agents, generating surplus from trading in each period while the agreement is in place. The long term agreement is dissolved by the decision of either party or by an exogenous shock. Upon dissolution of the long term relationship, the seller and the buyer return to the matching market. The objective function of each agent is the expected discounted average payoff.

We make the following three standard assumptions for the market for lemons. First, the cost of production of high quality good exceeds that of the low quality good as well as the benefit from the low quality good. Second, both the high quality good and the low quality good induce surplus, but the former is greater than the latter. Finally, the average benefit from the two types of good is lower than the production cost of the high quality good, the case in which the lemons problem is often called "severe."

We focus on a stationary equilibrium in which the agent's equilibrium threshold price depends only upon his type and whether or not he is in the long term relationship. We examine stationary equilibria in which trading occurs with a positive probability. We focus on the limit of a sequence of stationary equilibria as the search friction, quantified by the time span of each period, vanishes. We rigorously eliminate all frictional unemployment and frictional vacancy to see under what conditions involuntary unemployment and involuntary vacancy coexist persistently.

The main result of the present paper is a complete characterization of the conditions on the primitives under which coexistence arises. Roughly speaking, the more patient the agents are, the more likely coexistence arises. The patience of an agent is determined by his time preference as well as the probability that the long term relationship dissolves. It turns out that the ratio of the discount rate over the intensity of separation is a critical parameter to quantify the (im)patience of the agent. The higher the discount rate is relative to the intensity of separation, the more impatient are the agents.

Let us provide intuition for the main result in two steps. First, we state that our equilibrium is semi-pooling and give intuition behind this result. Second, we explain why the market clears if the agents are sufficiently impatient and why it does not otherwise.

To proceed with the first step, let us begin by considering the one-shot market with adverse selection of Akerlof (1970) as the benchmark. Since the average quality in the pool is below the production cost of the high quality good, no high quality seller is willing to trade. If trading occurs at all, only the low quality good will be traded, and the short side extracts the entire gain from trading. In particular, if the total demand is less than the total supply of the low quality good, then the buyer can extract positive surplus from knowingly purchasing a low quality good.

In the dynamic model, however, the static equilibrium outcome is no longer sustained by an equilibrium. To see this, suppose that the static market outcome is sustained as an equilibrium outcome. Suppose also that the mass of buyers is close to one (it could be either greater than, less than, or equal to one). If (almost) all the low quality sellers reach an agreement and leave the pool, then a buyer can infer by the end of the day that virtually all remaining sellers have a high quality good. Thus, if a price slightly above the production cost is offered, then both buyers and high quality sellers can reach an agreement. Contrary to what Akerlof (1970) predicts, a high quality seller can trade at a high price.

But, if a low quality seller can trade at a high price, the low quality seller has an incentive to wait for this to happen. Therefore, a high quality seller cannot reach an agreement at the high price "too" frequently in the equilibrium. If trade occurs at the high price frequently, then a low quality seller has an incentive to wait for this opportunity and refuse to trade at the lower price. Then due to the assumption that the lemons problem is severe, the average quality falls below the production cost of the high quality good, the high quality sellers stay out of the market, and trade does not occur at the high price. Thus, in the equilibrium, trade must occur at two prices: one at which only the low quality sellers are willing to trade, and the other at which both high and low quality sellers are willing to trade, i.e., Our equilibrium must be semi-pooling.[3]

Let us move to the second step to explain the role of impatience in market clearing. To begin with, the probability of reaching an agreement is determined by the bargaining position of a seller and a buyer. If the agents are sufficiently impatient, then the current match is more important than the future match. Therefore, given other things, the short side has more bargaining power than the long side. If the buyer is in the short side, a buyer tends to obtain a positive surplus, and is willing to reach agreement quickly.

Next, suppose that the agents are sufficiently patient (in the sense we defined above). We have shown in the first step that they cannot reach an agreement with a high probability at the higher of the two prices that trade occurs. At the lower of the two prices, the low quality seller sets the threshold high to decrease the probability of reaching an agreement since there is an opportunity of trading at the high price in the future. If the buyer's payoff is close to zero, the buyer incurs little cost of delaying in reaching an agreement, and therefore, there is little reason to hasten the agreement. In this case, the trading occurs so slowly that both parties have to stay in a market for a long time, which leads to coexistence. On the other hand, if the buyer's payoff is bounded away from zero, then he has a reason to give in and obtain a positive payoff today rather than in the future.

Adverse selection has been known to cause inefficiency of the equilibrium allocation. The outcome of Akerlof (1970) is inefficient because a low quality product is matched with a consumer, even though the efficiency requires that the high quality

[3]More precisely, trading occurs at two disjoint intervals of prices, each of which converges to a singleton as friction vanishes. See also Moreno and Wooders (2010) for a similar result.

product should be matched to a consumer. The reason for inefficiency is not the delay of reaching agreement between the high quality seller and a consumer, but the immediate match between a low quality seller and a buyer.

Moreno and Wooders (2010) is closely related to our paper. Both papers show that any stationary equilibrium must be semi-pooling where the high quality seller and the low quality seller are pooled at a higher price of the two equilibrium prices, and that the equilibrium outcome may entail significant delay in reaching agreement even in the limit as the friction vanishes.[4]

Our paper differs from Moreno and Wooders (2010) in a number of ways, as well. Moreno and Wooders (2010) is an "open" model where there is a constant flow of entry by the equal size of sellers and buyers in each period. On the other hand, our model is a "closed" model where there is no entry in the baseline model. In particular, the "closed" model allows us to consider the case in which the mass of buyers and the mass of sellers are different. As we admit more flexible configuration of demand and supply, our paper has a broader scope than Moreno and Wooders (2010). For example, under the assumption that the demand and the supply of goods are equal, Moreno and Wooders (2010) show that the delay of reaching agreement must persist, as friction vanishes. Our analysis replicates their result when the mass of buyers and the mass of sellers are equal. Moreover, we completely identify the conditions under which the delay in reaching agreement persists in the limit. That is, if the relative size of the mass of buyers to the mass of sellers is below a certain threshold, then the delay in reaching agreement vanishes, even though the resulting allocation is inefficient. Otherwise, there must be delay in reach agreement.

There are many works on one-sided involuntary unemployment or vacancy that remains even in the limit of friction vanishing. In Diamond (1971), a positive amount of excess supply persists in the equilibrium, but a buyer can purchase a good without any constraint. Shapiro and Stiglitz (1984) constructed a model in which, in the presence of moral hazard, the wage rate is set higher than the market clearing wage to induce involuntary unemployment of workers. In addition to reproducing the outcome of Shapiro and Stiglitz (1984) under adverse selection, our paper induces the coexistence of involuntary unemployment and involuntary vacancy and examines the condition under which this coexistence arises.[5]

Guerrieri, Shimer, and Wright (2010), Chang (2012), and Blouin and Serrano (2001) examined a class of matching models with adverse selection. A key assumption is that the matching technology is not efficient, in the sense that the short side in the matching pool may have a limited opportunity to meet a partner. Because

[4]Wolinsky (1990) is a seminal paper along this line of research. While Wolinsky (1990) considered a market for lemons with only two prices being admitted to choose by agents, Moreno and Wooders (2010) considered a similar situation in which price range is continuum.

[5]See also Stiglitz and Weiss (1981) and Azariadis (1975) for one-sided involuntary unemployment or vacancy.

both the inefficiency of matching technology and the informational asymmetry contribute to unemployment and vacancy, it is difficult to discern the two effects from one another or to identify how much of the unemployment and vacancy is caused by informational asymmetry, and how much is caused by the inefficiency of the matching technology. We choose an efficient matching technology so that the only source of friction other than informational asymmetry is the time span of each period, which we let vanish.

Matsui and Shimizu (2005) considered an infinitely repeated economy that has many marketplaces that agents choose to visit. They showed the existence of two price equilibria as well as single price ones. In the two price equilibria, the goods are traded at a high price in some marketplace where sellers are rationed, while they are traded at a low price in some other where buyers are rationed.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 presents the preliminary results and concepts. Section 4.2 formally describes the main results. Section 5 examines an important application of our main result to the model in which buyer can freely enter the market, while paying vacancy cost to stay in the matching pool. Section 6 considers a model in which true quality of the good is revealed dring the long term relationship. Section 7 examines a model with a take-it-or-leave-it offer. Section 8 concludes the paper.

2. Model

2.1. **Static model.** We consider an economy which is populated by 2 unit mass of infinitesimal (infinitely-lived) sellers, high type and low type sellers of equal size, and $x_b > 0$ unit mass of infinitesimal (infinitely-lived) buyers.[6]

High type sellers produce one unit of high quality good at the cost of c_h, while low type sellers produce one unit of low quality good at the cost of c_l. Assume $c_h > c_l$. The goods are indivisible. The marginal utility of the high quality good for a buyer is ϕ_h, while that of the low quality good is ϕ_l, where $\phi_h > \phi_l$. Each seller produces at most one unit of the good, and each buyer consumes at most one unit of the good.

We make the following three standard assumptions on the parameter values, which are critical for capturing the lemons problem.

- *A*1. $\phi_h > c_h > \phi_l > c_l$, which implies that the existence of the gains from trading under each state is common knowledge.
- *A*2. $\phi_h - c_h > \phi_l - c_l$ so that it is socially efficient for the high quality sellers to deliver the good to the buyers.

[6]No main result is qualitatively sensitive to the fact that the masses of high and low quality sellers are the same.

A3. $\dfrac{\phi_h + \phi_l}{2} < c_h$ so that the lemons problem is severe in the sense that random transactions lead to a negative payoff either to a buyer or to a high quality seller.

If p is the delivery price of the good, and $y(\in \{h, l\})$ is the quality of the good, seller's profit is $p - c_y$ and buyer's surplus is $\phi_y - p$. Under the assumptions we made, only the low quality good is traded in any competitive equilibrium, and the equilibrium price p^* is given by

$$p^* \in \begin{cases} \{c_l\} & \text{if } x_b < 1, \\ [c_l, \phi_l] & \text{if } x_b = 1, \\ \{\phi_l\} & \text{if } x_b > 1. \end{cases}$$

2.2. **Baseline model.** Let us embed the above static model into a decentralized dynamic trading model. We first describe the baseline model where the mass of buyers is exogenously fixed to $x_b > 0$.

Time is discrete, and the horizon is infinite. The time span of each period is $\Delta > 0$, which represents the amount of friction. When a buyer and a seller are initially matched at period t, conditioned on her type $k \in \{h, l\}$, the seller reports her type as $k' \in \{1, 2\}$, possibly in a randomized fashion, to a third party (or mechanism) which draws a price p according to a probability density function $f_{k'}$ over $\mathbb{R}$. We assume that the support of $f_{k'}$ is $[c_l, \phi_h]$.

We assume

$$\forall k' \in \{h, l\}, \ \forall p \in [c_l, \phi_h], \quad f_{k'}(p) > 0 \text{ and is continuous.} \tag{2.1}$$

Conditioned on p drawn by the mechanism, each party has to decide whether or not to form a long term relationship. After forming the long term relationship, the buyer can purchase the good at the agreed price, and the seller can sell the good at the same price to the buyer. If the good is delivered at p, the seller's surplus is $p - c_k$ and the buyer's surplus is $\phi_k - p$ $(k \in \{h, l\})$.

Then, at the end of the period, either one of two events will occur. The long term relationship breaks down with probability $1 - \delta$, and then, both agents are dumped back to the respective pools. The long term relationship continues with probability δ without the true quality being revealed.

In each period, the buyer and the seller in a long term relationship can choose to maintain or to terminate it. If one of the agents decides to terminate the long term relationship, both agents return to their respective pools, waiting for the next round of matching. If both agents decide to continue the long term relationship, the long term relationship continues with probability $\delta = e^{-d\Delta}$ where $d > 0$, and with probability $1 - \delta$, the long term relationship dissolves, and the two agents are forced to return to the pool.

We assume that the true quality of the good is not revealed to the buyer during the long term relationship, like a life insurance policy, until the long term relationship dissolves. This assumption is only to simplify exposition.[7]

The objective function of each agent is the long run discounted average expected payoff:

$$(1-\beta)\mathsf{E}\sum_{t=1}^{\infty}\beta^{t-1}u_{i,t}$$

where $u_{i,t}$ is the payoff of agent i in period t and $\beta = e^{-b\Delta}$ is the discount factor.

We focus on a simple class of equilibria, in which the equilibrium strategy of each player depends only upon the type and the status of the player: whether or not the player is in the long term relationship. A stationary equilibrium is a strategy profile where no player has an incentive to deviate, and a distribution of the agents in the pool does not change over time. Like many other bilateral trading models, this model admits a stationary equilibrium with no trading, as every player rejects every price following every history. We focus on the undominated stationary equilibrium, which is a stationary equilibrium where no dominated strategy is used, to exclude a "no trading equilibrium" in which every agent refuses to reach an agreement. We simply refer to an undominated stationary equilibrium as an equilibrium, whenever the meaning is clear from the context.

To simplify exposition, we assume for the rest of the paper that p is drawn from $[c_l, \phi_h]$ according to the uniform distribution regardless of the report of the seller. We can assume that p is drawn from $[c_l, \phi_h]$ according to a continuous density function $f(p) > 0\ \forall p \in [c_l, \phi_h]$, without changing any result, but only at the cost of significantly more cumbersome notation.[8]

3. Preliminaries

Let $W_s^h(p)$, $W_s^l(p)$, and $W_b(p)$ be the continuation values of a high quality seller, a low quality seller, and a buyer, respectively, after the two agents agree on $p \in [c_l, \phi_h]$. Also, let W_s^h, W_s^l, and W_b be the continuation values of respective agents after they do not form a long term relationship. Given the equilibrium value functions, let us characterize the optimal decision rule of each agent. In what follows, we write $x \le O(\Delta)$ if

$$\lim_{\Delta\to 0}\frac{x}{\Delta} < \infty.$$

Let z_s^l and z_s^h be the mass of c_l and c_h sellers in the pool. Similarly, let z_b be the mass of buyers in the pool. Since the mass of paired buyers and the mass of paired

[7]In subsection 6, we extend the model to the one in which the true quality is revealed to the buyer during the long term relationship with probability $1-\lambda$ per period, and upon the revelation of the true quality, the buyer can decide to continue or terminate the existing long term relationship.

[8]The extension to the case where the price is drawn from a general distribution satisfying (2.1) is (Cho and Matsui (2013)).

sellers are of equal size, we have

$$2 - z_s = x_b - z_b, \tag{3.2}$$

where $z_s = z_s^h + z_s^l$. Let

$$\mu_h = \frac{z_s^h}{z_s}$$

be the proportion of high quality sellers in the pool of sellers, and let $\mu_l = 1 - \mu_h$ be the proportion of low quality sellers in the pool of sellers.

Our first goal is to find conditions under which

$$\lim_{\Delta \to 0} z_b > 0,$$

and

$$\lim_{\Delta \to 0} z_s > 0$$

hold simultaneously in the baseline model, where the mass of buyers is fixed. Throughout the paper, z_s is interpreted as (involuntary) unemployment, while z_b as (involuntary) vacancy.

Because the relative size of buyers and sellers in the pool is an important variable, let us define

$$\rho_{bs} = \frac{z_b}{z_s}.$$

Since ρ_{bs} determines the frequency of meeting the other party with a long side rationed, let us define

$$\zeta = \min\{1, \rho_{bs}\}$$

as the probability that a seller meets a buyer, and

$$\xi = \min\left\{1, \frac{1}{\rho_{bs}}\right\} \tag{3.3}$$

as the probability that a buyer meets a seller, where we treat $1/0 = \infty$. Due to (3.2), we have

$$\begin{cases} \zeta = \rho_{bs} < 1 \text{ and } \xi = 1 & \text{if } x_b < 2, \\ \zeta = \rho_{bs} = 1 \text{ and } \xi = 1 & \text{if } x_b = 2, \\ \zeta = 1 \text{ and } \xi = \frac{1}{\rho_{bs}} < 1 & \text{if } x_b > 2. \end{cases}$$

Let Π_s^h be the set of prices that a high quality seller and a buyer agree to accept, and let $\pi_s^h = \mathsf{P}(\Pi_s^h)$. For $p \in \Pi_s^h$, we can write

$$W_s^h(p) = (1-\beta)(p - c_h) + \beta\left(\delta W_s^h(p) + (1-\delta) W_s^h\right).$$

The first term is the payoff in the present period. At the end of the present period, with probability $1-\delta$, the long term relationship dissolves, and the high quality seller's continuation payoff is W_s^h. With probability δ, the high quality seller continues the relationship, of which continuation value is given by $W_s^h(p)$.

A simple calculation shows

$$W_s^h(p) = \frac{(1-\beta)(p-c_h) + \beta(1-\delta)W_s^h}{1-\beta\delta}. \tag{3.4}$$

The high quality seller agrees to form a long term relationship with delivery price p if

$$W_s^h(p) > W_s^h$$

which is equivalent to

$$p > c_h + W_s^h. \tag{3.5}$$

On the other hand, W_s^h is given by

$$W_s^h = \beta\zeta\pi_s^h \mathsf{E}[W_s^h(p)|\Pi_s^h] + \beta(1-\zeta\pi_s^h)W_s^h. \tag{3.6}$$

Substituting (7.36) into (7.37), we obtain, after some calculation,

$$W_s^h = \frac{\beta\zeta\pi_s^h}{1-\beta\delta}\mathsf{E}[p - c_h - W_s^h|\Pi_s^h]. \tag{3.7}$$

Similarly, we obtain

$$W_s^l = \frac{\beta\zeta\pi_s^l}{1-\beta\delta}\mathsf{E}[p - c_l - W_s^l|\Pi_s^l], \tag{3.8}$$

where Π_s^l is the set of prices that a low quality seller and a buyer agree to accept, and $\pi_s^l = \mathsf{P}(\Pi_s^l)$. In any undominated equilibrium, c_l seller accept p if

$$p > c_l + W_s^l.$$

Imitating the behavior of high quality sellers, a low quality seller can always obtain a higher (or equal) continuation value than a high quality seller.[9] Therefore, we have $W_s^l \geq W_s^h$. Now, we would like to claim that the threshold price for a low quality seller is lower than that for a high quality seller.

Lemma 3.1.

$$c_h - c_l > W_s^l - W_s^h.$$

Proof. If a high quality seller imitates a low quality seller, then the long run expected payoff from the deviation is

$$W_s^l - (c_h - c_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l}.$$

Since the deviation payoff is less than the equilibrium payoff,

$$W_s^l - W_s^h \leq (c_h - c_l)\frac{\beta\pi_s^l}{1-\beta\delta+\beta\pi_s^l} < c_h - c_l$$

as desired. □

[9] If the true quality is revealed with a positive probability after the good is delivered, then we cannot invoke the same argument to prove the inequality. Yet, the main result is carried over.

Let Π_s^l (resp. Π_s^h) be the set of prices where L-type (resp. H-type) sellers and buyers trade with a positive probability. Lemma 3.1 says

$$c_l + W_s^l = \inf \Pi_s^l < c_h + W_s^h = \inf \Pi_s^h.$$

Since the decision rule of each seller is a threshold rule, this inequality implies

$$\Pi_s^h \subset \Pi_s^l.$$

Thus, we can partition the set of prices into three regions, Π_s, Π_p, and the rest:

$$\begin{aligned} \Pi_s &= \Pi_s^l \setminus \Pi_s^h, \\ \Pi_p &= \Pi_s^l \cap \Pi_s^h, \end{aligned}$$

where Π_s is the set of the prices at which trade occurs only with low quality sellers (the subscript stands for separating), Π_p is the set of the prices at which trade occurs with both low and high quality sellers (the subscript stands for pooling), and the remaining region is the one in which no trade occurs. Note that we have

$$\begin{aligned} \Pi_s &\subset [c_l + W_s^l, c_h + W_s^h], \\ \Pi_p &\subset [c_h + W_s^h, \infty). \end{aligned}$$

We shall focus on a stationary equilibrium in which the strategy of each player depends only upon the status of the player, i.e., whether or not he is in the pool or in the long term relationship. As in most bilateral trading models, this game admits a stationary equilibrium, in which each player rejects all prices, following every history. In this equilibrium, also known as no trading equilibrium, some players have to use (weakly) dominated strategies. Note that the equilibrium payoff is 0 for every player. Under $A1$, any price $p \in (c_l, \phi_l)$ is accepted, if all, only by the low quality seller. If the buyer accepts such p, he can still generate strictly positive surplus. Only because each player reject such p with probability 1, it is optimal to reject p. If p is accepted by the other party with a positive probability, it is the best response to accept p. Thus, rejecting p is a (weakly) dominated strategy.

Eliminating no trading equilibrium, we focus on a stationary equilibrium in which trading occurs with a positive probability. Let $\pi_s = \mathsf{P}(\Pi_s)$ and $\pi_p = \mathsf{P}(\Pi_p)$. Since we focus on an equilibrium in which trading occurs with a positive probability,

$$\pi_s + \pi_p > 0$$

in an equilibrium. Since we eliminate no trading equilibrium, which involves (weakly) dominated strategies, we call our equilibrium an undominated stationary equilibrium, or simply, an equilibrium, whenever the meaning is clear from the context.

Definition 3.2. *If $\pi_p = 0$ in an equilibrium, we call such an equilibrium a separating equilibrium. If $\pi_s = 0$, then the equilibrium is called a pooling equilibrium. If $\pi_s > 0$ and $\pi_p > 0$, then it is called a semi-pooling equilibrium.*

Let us calculate the value function of a buyer. In the private value model in which a buyer knows exactly how valuable the objective is (Cho and Matsui (2017)), the informational content of p is irrelevant for a buyer to deciding whether or not to accept p. In contrast, in the present model, the expected quality conditioned on p is a critical factor for a buyer to make a decision.[10] Let $\phi^e(p)$ be the expected quality if p is the price to be agreed upon. If $p \in (c_l + W_s^l, c_h + W_s^h)$, then only low quality sellers agree to accept the price, and therefore, we have $\phi^e(p) = \phi_l$. On the other hand, if $p > c_h + W_s^h$ holds, then both low and high quality sellers agree to do so, and therefore, we have

$$\phi^e(p) = \phi(\mu_l) \equiv \mu_l\phi_l + (1-\mu_l)\phi_h.$$

If a buyer and a seller agree to form a long term relationship at price p, then the expected continuation value of the buyer is given by

$$W_b(p) = (1-\beta)(\phi^e(p) - p) + \beta\left[\delta W_b(p) + (1-\delta)W_b\right].$$

Therefore, we have

$$W_b(p) = \frac{(1-\beta)(\phi^e(p) - p) + \beta(1-\delta)W_b}{1-\beta\delta}.$$

Also, the continuation value after no match is given by

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[W_b(p)|\Pi_p\right] + \beta(1-\xi\mu_l\pi_s - \xi\pi_p)W_b. \tag{3.9}$$

After substitutions and tedious calculation, we obtain

$$W_b = \frac{\beta\xi\mu_l\pi_s}{1-\beta\delta}\mathsf{E}\left[\phi_l - p - W_b|\Pi_s\right] + \frac{\beta\xi\pi_p}{1-\beta\delta}\mathsf{E}\left[\phi(\mu_l) - p - W_b|\Pi_p\right] \tag{3.10}$$

where ξ is the probability that a buyer is matched to a seller, as defined in(3.3).

A buyer is willing to accept p if

$$W_b(p) > W_b,$$

or equivalently,

$$\phi^e(p) - p > W_b.$$

Since $\phi^e(p)$ may change as p changes, the buyer's equilibrium decision rule may not be characterized by a single threshold.

[10]Even if each individual is infinitesimally small, the informational content of p affects the decision of all buyers. In this sense, each individual is not "informationally small" in the sense of Gul and Postlewaite (1992).

Combining these results and including the endpoints as they are measure zero events, we have

$$\Pi_s = \begin{cases} [c_l + W_s^l, \phi_l - W_b] & \text{if } c_l + W_s^l \leq \phi_l - W_b, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\Pi_p = \begin{cases} [c_h + W_s^h, \phi(\mu_l) - W_b] & \text{if } c_h + W_s^h \leq \phi(\mu_l) - W_b, \\ \emptyset & \text{otherwise.} \end{cases}$$

By the assumption that p is uniformly distributed over (c_l, ϕ_h), we obtain

$$\mathsf{E}[p - c_h - W_s^h | \Pi_s^h] = \mathsf{E}[p - c_h - W_s^h | \Pi_p] = A\pi_p,$$

where

$$A = \frac{1}{2}(\phi_h - c_l).$$

Therefore, (7.38) can be rewritten as

$$W_s^h = \frac{\beta A (\pi_p)^2 \zeta}{1 - \beta\delta}. \tag{3.11}$$

Similarly, we have

$$\mathsf{E}[p - c_l - W_s^l | \Pi_s] = A\pi_s, \tag{3.12}$$

$$\mathsf{E}[\phi(\mu_l) - p - W_b | \Pi_p] = A\pi_p, \tag{3.13}$$

$$\mathsf{E}[\phi_l - p - W_b | \Pi_s] = A\pi_s. \tag{3.14}$$

Thus, W_s^l and W_b can be rewritten as

$$W_s^l = \frac{\beta A (\pi_s)^2 \zeta}{1 - \beta\delta} + \frac{\beta \pi_p \zeta}{1 - \beta\delta} \mathsf{E}[p - c_l - W_s^l | \Pi_p] \tag{3.15}$$

$$W_b = \frac{\beta A (\pi_s)^2 \mu_l \xi}{1 - \beta\delta} + \frac{\beta A (\pi_p)^2 \xi}{1 - \beta\delta}, \tag{3.16}$$

respectively. Following Cho and Matsui (2013), one can obtain a similar expression for a general distribution of price, given a sufficiently small $\Delta > 0$.

Also, rewrite π_s and π_p as

$$\pi_s = C[\phi_l - c_l - W_b - W_s^l] \tag{3.17}$$

$$\pi_p = C[\phi(\mu_l) - c_h - W_b - W_s^h] \tag{3.18}$$

where

$$C = \frac{1}{\phi_h - c_l}.$$

The size of population of each type of the agents is determined by the balance equations:

$$1 - z_s^l = \left(\frac{\pi_s \zeta}{1-\delta} + \frac{\pi_p \zeta}{1-\delta}\right) z_s^l \tag{3.19}$$

$$1 - z_s^h = \frac{\pi_p \zeta}{1-\delta} z_s^h \tag{3.20}$$

$$x_b - z_b = \left(\frac{\pi_s \mu_l \xi}{1-\delta} + \frac{\pi_p \xi}{1-\delta}\right) z_b. \tag{3.21}$$

An equilibrium in the main model is characterized by $(z_b, z_s^h, z_s^l, W_b, W_s^l, W_s^h)$. We use the following notion of market clearing.

Definition 3.3. *A market fails to clear if (involuntary) unemployment and (involuntary) vacancy coexist:*

$$\lim_{\Delta \to 0} z_b z_s = \lim_{\Delta \to 0} z_b (z_s^h + z_s^l) > 0. \tag{3.22}$$

Otherwise, the market clears.

Our goal is to identify the condition under which adverse selection prevents the market from clearing itself.

4. Analysis

Since the main result takes a number of steps, it will be helpful to illustrate the reasoning process toward the main result.

4.1. **Overview.** Note that z_b, z_s^h, z_s^l are functions of π_s and π_p. The smaller π_s and π_p are, the larger z_b, z_s^h, z_s^l. As $\Delta \to 0$, each player has more opportunities to meet his potential partner for a given amount of (real) time. The ensuing analysis shows that $\pi_s, \pi_p \to 0$ as $\Delta \to 0$.

In order to identify the condition under which (3.22) occurs, we need to analyze how quickly $\pi_s, \pi_p \to 0$ as $\Delta \to 0$. To be concrete, suppose that π_s, π_p vanishes at a slower rate than Δ. Then, for $\forall \tau > 0$ amount of real time, the agreeable prices arrive quickly so that all opportunities to trade will be exhausted as $\Delta \to 0$. If so, the market must clear. On the other hand, if π_s, π_p vanishes at the rate of Δ, then the trading occurs slowly enough so that some traders have to remain in the pool for a significant amount of time before reaching agreement. In fact, we are looking for the condition in the baseline model where $\pi_s, \pi_p \to 0$ at the rate of $\Delta > 0$.

The question is, then, what causes $\pi_s, \pi_p \to 0$ "quickly." Remember that $\Pi_s = [c_l + W_s^l, \phi_l - W_b]$ and $\Pi_p = [c_h + W_s^h, \phi^e - W_b]$. The lower bound of Π_s is determined by the equilibrium threshold price of c_l seller, while its upper bound is the equilibrium threshold price of a buyer. The size of Π_s, and π_s is therefore determined by how each party sets the threshold.

The first crucial step is to identify the source of the bargaining power of each player. Note that c_l seller has an option to sell at a price in Π_s, but also at a price in Π_p. If trading occurs at a price in Π_s, the gain from trading over W_s^l is small, since Π_s is shrinking to a single point as $\Delta \to 0$. However, if trading occurs at a price in Π_p, a c_l seller can generate at least $c_h - \phi_l > 0$. Thus, c_l seller sets the threshold to make π_s small.

If the buyer sets the threshold sufficiently higher than that of the c_l seller, then trading can occur in Π_s frequently to clear the market. The next crucial step is to identify the condition under which a buyer is willing to set the threshold price close to that of the c_l seller. A buyer is willing to trade frequently if delay is costly, which is the case if $\lim_{\Delta\to 0} W_b > 0$. If $W_b \to 0$, however, a buyer has little to lose by delaying the agreement. Theorem 4.5 completely characterizes the condition under which $\lim_{\Delta\to 0} W_b = 0$ in the baseline model under which the market fails to clear.

4.2. **Results.** Let us state the asymptotic properties of the equilibrium payoffs for the case where $A1 - A3$ hold.

Proposition 4.1. *For any sequence of undominated stationary equilibria,*

$$\begin{aligned} \lim_{\Delta\to 0} W_s^h &= 0 \\ \lim_{\Delta\to 0} W_s^l + W_b &= \phi_l - c_l \end{aligned}$$

Proof. See Appendix A. □

In order to understand how the equilibrium surplus $\phi_l - c_l$ is split between a seller and a buyer, we need to investigate the structure of an equilibrium further. The next lemma is a critical step toward characterizing the condition under which the market fails to clear.

Lemma 4.2.

$$\lim_{\Delta\to 0} \zeta W_b = 0.$$

Proof. See Appendix B. □

Lemma 4.2 reveals the complementary slackness between ζ and W_b in the limit as $\Delta \to 0$. Note that except for the knife-edge case of $x_b = 2$, $\zeta > 0$ is equivalent to $z_b > 0$. Suppose that W_b is bounded away from zero, and $z_b > 0$. If $W_b > 0$, the delay of reaching agreement is costly for the buyer. Thus, buyers have incentive to accelerate the trade by increasing the equilibrium threshold slightly to accept higher price, to realize the gain from trading before other buyers do. This logic continues to hold, as long as $W_b > 0$ and buyers are competing among themselves for the opportunites for trading. Thus, in an equilibrium, W_b or z_b converge to zero as friction vanishes.

The next proposition states that $z_b > 0$ in the limit implies $z_s > 0$. The coexistence of involuntary unemployment and vacancy arises if and only if $z_b > 0$ holds in the limit as Δ goes to zero. From the one-to-one nature of the matching technology, this proposition reveals the critical condition under which z_b is bounded away from 0.

Lemma 4.3. *If*
$$\lim_{\Delta\to 0} z_b > 0,$$
then
$$\lim_{\Delta\to 0} z_s = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} \in (0,2). \tag{4.23}$$

Proof. From (3.19) and (3.20), we have
$$z_s^h = \frac{1}{1 + \frac{\zeta\pi_p}{1-\delta}}$$
and
$$z_s^l = \frac{1}{1 + \frac{\zeta\pi_p + \zeta\pi_s}{1-\delta}}.$$

We use the following lemma for the proof of Lemma 4.3.

Lemma 4.4. *Suppose that* $\lim_{\Delta\to 0} z_b > 0$. *Then*
$$\lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\delta} = Q_p \equiv \frac{b+d}{d}\left[\frac{\phi_l - c_l}{c_h - \phi_l}\right], \tag{4.24}$$
$$\lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\delta} = Q_s \equiv \left[\frac{2c_h - (\phi_h + \phi_l)}{\phi_h - c_h}\right](1 + Q_p). \tag{4.25}$$

Proof. See Appendix C. □

Lemma 4.4 implies
$$\lim_{\Delta\to 0} z_s^h = \frac{1}{1+Q_p} = \frac{c_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}$$
and
$$\lim_{\Delta\to 0} z_s^l = \frac{1}{1+Q_s+Q_p} = \frac{\phi_h - c_h}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}$$
which are independent of x_b. Thus, if $\lim_{\Delta\to 0} z_b > 0$, then
$$0 < \lim_{\Delta\to 0} z_s = \lim_{\Delta\to 0} z_s^h + z_s^l = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)}. \tag{4.26}$$

Under $A1 - A3$, one can easily verify that
$$\phi_h - \phi_l < 2(c_h - c_l)$$

from which

$$\frac{\phi_h - \phi_l}{c_h - c_l + \dfrac{b}{d}(\phi_l - c_l)} < 2 \tag{4.27}$$

follows. □

Note that the right hand side of (4.23) is independent of z_b. $\lim_{\Delta\to 0} z_s$ is independent of how many buyers are in the pool, or whether or not buyers are on the short side, as long as a positive mass of buyers are in the pool.

Our goal is to identify the conditions under which the market fails to clear:

$$\lim_{\Delta\to 0} z_b z_s > 0.$$

Thanks to Lemma 4.3, it suffices to completely characterize the conditions under which $\lim_{\Delta\to 0} z_b > 0$ holds, which in turn implies that the market fails to clear, i.e.,

$$\lim_{\Delta\to 0} z_b z_s > 0.$$

Theorem 4.5. *The market fails to clear, i.e.,*

$$\lim_{\Delta\to 0} z_b z_s > 0$$

if and only if we have

$$\frac{\phi_h - \phi_l}{c_h - c_l + \dfrac{b}{d}(\phi_l - c_l)} - (2 - x_b) > 0. \tag{4.28}$$

In particular, the market fails to clear whenever $x_b \geq x_s = 2$.

We have shown that $z_s > 0$ if $z_b > 0$. If $x_b > x_s = 2$, $z_b > 0$ automatically holds. Therefore, the coexistence of involuntary unemployment and involuntary vacancy arises without fail. We also have a complementary slackness condition, $\zeta W_b = 0$ in the limit as Δ goes to zero where ζ is the probability that a seller meets a buyer per period. Except for a knife-edge case of $x_b = x_s$, $\zeta > 0$ if and only if $z_b > 0$. In other words, $z_b > 0$, then $W_b = 0$.

If $x_b > 2$, then (4.28) holds. In fact, (4.28) holds if and only if $W_b = 0$ in the limit as the friction disappears. In this case, we have shown that the delay of reaching an agreement must arise, and the coexistence of the involuntary unemployment and the involuntary vacancy arises. One might ask why the buyer cannot accelerate the trading by increasing the equilibrium threshold price. The reason is exactly the same as we discussed in the context of Moreno and Wooders (2010). Since the buyer's payoff is close to 0, he cannot raise the price without violating his own individual rationality. An immediate consequence of binding individual rationality is that it is optimal for the buyers to delay in reaching an agreement as the cost of delay becomes very small.

On the other hand, if $x_b < 2$, then $z_b > 0$ may or may not hold as one can see it from the threshold condition (4.28). Still, in this case, $z_s > 0$ continues to hold. As long as (4.28) holds, the individual rationality of the buyer is binding, and the same argument applies to show that the coexisting of involuntary unemployment and the involuntary vacancy arises.

In order to better understand the precise role of the binding individual rationality of the buyer, let us examine the case where (4.28) is violated so that $W_b > 0$ even in the limit as $\Delta \to 0$. If $x_b < x_s = 2$ and if the agents are sufficiently impatient so that b/d is sufficiently large. The more impatient the agents become, the more incentive they have to speed up the trade. Recall that $z_s > 0$ whenever $x_b < x_s = 2$. If the buyer reaches an agreement quickly, then the buyer's side in the pool becomes very thin, i.e., $z_b \to 0$ as $\Delta \to 0$. As a result, a seller has to stay in the pool before a seller is matched to a buyer. The low quality seller always generates a positive surplus from trading, and therefore, the delay is particularly costly. Thus, the low quality seller is willing to lower the equilibrium threshold in order to accelerate the sales. As a result, the buyer can purchase the good at a lower price, which leads to the positive surplus in the long run. This is exactly what happens if (4.28) is violated. Since the buyers leave the pool quickly, no involuntary vacancy exists, but only the involuntary unemployment arises if (4.28) is violated.

Combined with Lemma 4.2, we conclude that the buyer's equilibrium payoff must vanish, whenever the market fails to clear.[11] Note that (4.28) can hold even if the mass of buyers is smaller ($x_b < 1$) than the mass of low quality sellers, where Akerlof (1970) predicts that the buyer should receive all equilibrium surplus from trading. We show otherwise, completely characterizing the condition under which the prediction from the static model is carried over to a dynamic model.

Observe that given other things, (4.28) will fail if the agents are very impatient so that b/d is large. For example, if $b/d = \infty$ and (4.28) fails, our model is essentially identical with the static model of Akerlof (1970), and the market clears in the sense that $\lim_{\Delta \to 0} z_b = 0$ when the buyers are on short side. The substance of Theorem 4.5 is to show that the intuition of Akerlof (1970) is carried over, as long as the agents are impatient in the sense that b/d is large.

The low quality seller can generate a large profit by agreeing on $p \in \Pi_p$ even if $\Delta > 0$ is small. However, trading at a high price from Π_p can be realized after possibly many rounds of matching and bargaining. If $b > 0$ is large so that (4.28) fails, then the seller is too impatient to exploit the future opportunity of trading at a high price, and is content with reaching an agreement quickly, which leads to $\lim_{\Delta \to 0} z_b = 0$, as Theorem 4.5 implies.

[11] The reverse is also true. Proposition D.1 says that if the market clears in the limit ($\lim_{\Delta \to 0} z_b = 0$), then the buyer's equilibrium payoff remains positive and (4.28) must be violated.

Proof. We state the proof for the necessity, while relegating the proof for the sufficiency to Appendix D. Suppose that

$$\lim_{\Delta\to 0} z_b > 0.$$

By Lemma 4.3, (4.23) holds. Since the matching is one to one,

$$z_b = x_b - 2 + z_s.$$

Substituting z_s by (4.23), we have

$$0 < \lim_{\Delta\to 0} z_b = \frac{\phi_h - \phi_l}{c_h - c_l + \frac{b}{d}(\phi_l - c_l)} - (2 - x_b). \tag{4.29}$$

which implies (4.28) holds. □

The intuition for (4.28) is as follows.[12] It is helpful to differentiate two types of probability of trading. By π_p and π_s, we mean the probabilities that trading physically occurs per period. On the other hand, by the *discounted sum* of the probabilities, we mean the probability that the trading can ever occur in the game.

First, like in other dynamic trading models with severe adverse selection, the maximal discounted sum of the trading probabilities of the c_h type sellers must be bounded away from 1. Otherwise, the c_l type sellers would have incentives to mimic c_h type sellers and can trade goods at a price greater than or equal to c_h with probability 1 as well, violating the buyers' individual rationality constraint implied by $A3$. Given the discounted sum of the trading probabilities for the c_h type sellers, π_p decreases as the agents become more patient, i.e., b/d decreases.

Second, the discounted sum of the trading probabilities of the c_l type sellers must be bounded away from 1. Otherwise, the fraction of the c_l type sellers becomes negligible, and the average quality of goods in the pool converges to ϕ_h. Then the buyers are willing to accept the high prices with a high probability since the lemon's problem vanishes. This is a contradiction to the first step. For the same reason as in the case of the c_h type sellers, given the discounted sum of the trading probabilities for the c_l type sellers, π_s decreases as the agents become more patient, i.e., b/d decreases.

Finally, given these upper bounds on the trading probabilities, there exists an upper bound less than 2, which is the total mass of the sellers, on the number of buyers who can trade in equilibrium. Therefore, if the number of buyers exceeds some threshold less than 2, not all buyers can trade, because the physical probability of trading per period becomes so small. Hence, the smaller b/d is, the likelier (4.28) is to hold.

[12] We are grateful for an anonymous referee for the precise and clear intuition, especially about the link between the patience and the quality of the product. We chose to follow the statement of the referee as closely as possible.

Theorem 4.1 shows that the coexistence of involuntary unemployment and involuntary vacancy can arise for a broad range of x_b, as long as (4.28) holds. In particular, even if $x_b > 2$ so that the sellers are in the short side, Theorem 4.1 implies that $z_s > 0$, as long as (4.28) holds. Note that if b/d is very small, (4.29) can hold, even if x_b is slightly less than 2. Even though the buyers are in the short side, the low quality seller is so patient that he sets the equilibrium threshold to accept only a very small portion of randomly generated prices. As the rate of reaching agreement slows down, some buyers have to stay in the market for an extended amount of time, driving down the equilibrium payoff of the buyer to zero, as $\Delta \to 0$.

5. Model of free entry

In the baseline model, $x_b > 0$ is an exogenous parameter. Vast majority of the models in the existing literature assume some form of free entry of agents (e.g., Rubinstein and Wolinsky (1985)). If we allow both buyers and sellers to enter the market to replace those who leave the market after reaching agreement, the total supply and demand become infinite. As a result, it is not clear how to formalize the notion of the unemployment rate and the vacancy rate, which are the key concepts to formulate the market clearing condition.

Let us consider the case that buyers can enter the market freely, while a buyer has to pay vacancy cost $F > 0$ per unit of time, or ΔF per period while remaining in the pool, as in Mortensen and Pissarides (1994). Also, as in Mortensen and Pissarides (1994), if a buyer and a seller dissolve the existing long term relationship, then the buyer permanently exits from the economy, while the seller returns to the pool. Since the mass of sellers is fixed, we can naturally define the unemployment rate.

Due to the free entry condition,

$$W_b = 0 \tag{5.30}$$

must hold in any equilibrium. Due to the assumption of job destruction together with (5.30), we have

$$W_b = \beta\xi\mu_l\pi_s \mathsf{E}\left[W_b(p)|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[W_b(p)|\Pi_p\right] - (1-\beta)\Delta F = 0. \tag{5.31}$$

Since a buyer exits permanently after the existing long term relationship dissolves,

$$W_b(p) = \frac{1-\beta}{1-\beta\delta}(\phi^e(p) - p) \tag{5.32}$$

where $\phi^e(p)$ is the expected quality conditioned on reaching agreement at p.

Substituting (5.32) into (5.31) and noticing the properties of Π_s and Π_p, we obtain

$$\beta\xi\mu_l\pi_s \mathsf{E}\left[\frac{\phi_l - p}{1-\beta\delta}|\Pi_s\right] + \beta\xi\pi_p \mathsf{E}\left[\frac{\phi(\mu_l) - p}{1-\beta\delta}|\Pi_p\right] = \Delta F. \tag{5.33}$$

Substituting (3.13) and (3.14) with $W_b = 0$ into (5.33), we obtain

$$\frac{\beta A(\pi_s)^2 \mu_l \xi}{1-\beta\delta} + \frac{\beta A(\pi_p)^2 \xi}{1-\beta\delta} = \Delta F. \tag{5.34}$$

We claim that ξ is uniformly bounded away from zero. Suppose not, i.e., $\xi = O(\Delta^\alpha)$ for some $\alpha > 0$. Then (5.34) implies $(\pi_p)^2 = O(\Delta^{2-\alpha})$. From (3.15), W_s^l goes to infinity as Δ goes to zero, which is a contradiction. Next, using the same argument as in Lemma B.2, we can show that both π_s and π_p converge to zero at the same rate. Thus, in order to balance the rates of convergence between the left and right hand sides of (5.34), we must have $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. This implies, from the analysis of the baseline model, that $\lim_{\Delta\to 0} z_s > 0$ and $\lim_{\Delta\to 0} z_b > 0$ must hold. The market fails to clear.

Theorem 5.1. *The market always fails to clear, i.e.,*

$$\lim_{\Delta\to 0} z_b z_s > 0.$$

Regarding the failure to clear the market, the same intuition as in the baseline model applies here. Indeed, the buyers have no strict incentive to increase the speed of reaching agreement by further decreasing the surplus, which violates the individual rationality constraint. Unlike in the baseline model, since $W_b = 0$ is guaranteed in the free entry model, the failure to clear the market necessarily occurs.

6. Revelation of quality

To simplify notation, we have assumed so far that the true quality of the good is not revealed until the existing long term relationship is dissolved. In order to understand how the information revelation affects the equilibrium outcome, suppose that a buyer and a seller are in the long term relationship, who have agreed to deliver one unit of the good from the seller to the buyer at price p. After the good is delivered to the buyer, the true quality is revealed with probability $1-\lambda = 1-e^{-\Delta\theta}$ ($\theta > 0$). Based upon the available information about the good, if any, the buyer and the seller decide whether to continue the long term relationship or not. If both agents decide to continue the long term relationship, then the two agents remain in the same relationship with probability $\delta = e^{-\Delta d}$. Even if both agents choose to stay in the long term relationship, with probability $1-\delta$, the relationship is dissolved immediately, and the two agents return to their respective pools. If either agent decides to terminate the long term relationship, then the relationship is dissolved immediately and the two agents return to the respective pools. The rest of the rules of the game remain the same.

An important implication of the new information is that the buyer has an option to terminate the long term relationship, if he discovers the quality is low, and to continue the relationship, if the quality is high. While the new information allows the buyer to get rid of low quality goods, the ensuing analysis reveals that as long as the lemon's problem is severe, the results in the previous section are carried over.

Since the new information arrives in each period with a positive probability, however, we need to modify assumption $A3$ accordingly:

$A3'$. The lemons problem is severe in the sense that

$$\frac{\phi_h + \phi_l}{2} + \frac{1}{2}\frac{\theta}{b+d}\phi_h < c_h.$$

The first term of the left hand side is the average quality of the good when the good is purchased. After the good is purchased, the true quality is revealed with probability $1 - e^{-\Delta\theta}$, while the agent discounts the future payoff at the rate of $e^{-\Delta b}$, and the long term relationship lasts with a probability of $e^{-\Delta d}$. After the true quality is revealed, only the high quality goods will be kept, which make up one half of the goods purchased by the buyer. The second term is the expected average discounted quality, conditioned on the event that the quality is revealed, and only the high quality good is kept.

Purchasing a good has an option value of observing the true quality, in addition to consuming the average quality. A tedious calculation shows that if price p is sufficiently high so that both high and low quality sellers agree to sell the good, the buyer accepts p when

$$\tilde{\phi}^e - p \geq W_b$$

where

$$\tilde{\phi}^e = \frac{\mu_l\phi_l + (1-\mu_l)\phi_h + \dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}\phi_h}{1 + \dfrac{\beta(1-\lambda)(1-\mu_l)}{1-\beta\delta}}.$$

Define

$$\Pi_p = [c_h + W_s^h, \tilde{\phi}^e - W_b] \qquad \text{and} \qquad \Pi_s = [c_l + W_s^l, \phi_l - W_b].$$

Then, we can calculate the value of each type of the agent conditioned on the event that he is in the pool:

$$W_s^h = \frac{\zeta\beta\pi_p \mathsf{E}(p - c_h - W_s^h \mid \Pi_p)}{1 - \beta + \beta\lambda(1-\delta)},$$

$$W_s^l = \frac{\zeta\beta\pi_s \mathsf{E}(p - c_l - W_s^l \mid \Pi_s)}{1 - \beta\lambda\delta} + \frac{\xi\beta\pi_p \mathsf{E}(p - c_l - W_s^l \mid \Pi_p)}{1 - \beta\lambda\delta},$$

and

$$W_b = \frac{\xi\beta\mu_l\pi_s}{1-\beta\delta}\mathsf{E}(\phi_l - p - W_b \mid \Pi_s) + \frac{\zeta\beta\pi_p}{1-\beta\lambda\delta}\left(1 + \frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right)\mathsf{E}(\tilde{\phi}^e - p - W_b \mid \Pi_p).$$

These values can be rewritten in a form more convenient for the analysis if the price is drawn from a uniform distribution:

$$W_s^h = A\frac{\zeta\beta\pi_p^2}{1-\beta+\beta\lambda(1-\delta)},$$

$$W_s^l = A\frac{\zeta\beta\pi_s^2}{1-\beta\lambda\delta} + \frac{\zeta\beta\pi_p \mathsf{E}(p-c_l-W_s^l \mid \Pi_p)}{1-\beta\lambda\delta},$$

and

$$W_b = A\frac{\xi\beta\mu_l\pi_s^2}{1-\beta\delta} + A\frac{\xi\beta\pi_p^2}{1-\beta\lambda\delta}\left(1+\frac{\beta(1-\lambda)(1-\delta)}{1-\beta\delta}\right).$$

Along with the balance equations, we can solve for the equilibrium outcome $(z_b, z_s^h, z_s^l; W_b, W_s^h, W_s^l)$. We are interested in the case where

$$\lim_{\Delta\to 0} z_b > 0.$$

If

$$\lim_{\Delta\to 0} z_b > 0,$$

then $\lim_{\Delta\to 0} W_b = 0$ and $\lim_{\Delta\to 0} W_s^h = 0$ imply

$$\lim_{\Delta\to 0} \tilde{\phi}^e - c_h = 0.$$

Thus, we have

$$\lim_{\Delta\to 0} \mu_l = \frac{\left(1+\frac{\theta}{b+d}\right)\phi_h - c_h}{\left(1+\frac{\theta}{b+d}\right)\phi_h - \phi_l}.$$

We need to modify (4.28) accordingly:

$$\frac{\left(1+\frac{\theta}{b+d}\right)\phi_h - \phi_l}{c_h - c_l + \dfrac{b}{d}(\phi_l - c_l)} - (2-x_b) > 0$$

which is a sufficient and necessary condition for

$$\lim_{\Delta\to 0} W_b = 0.$$

Summarizing the above, we have the following theorem.

Theorem 6.1. *The market fails to clear, i.e.,*

$$\lim_{\Delta\to 0} z_b z_s > 0$$

if and only if we have

$$\frac{\left(1+\frac{\theta}{b+d}\right)\phi_h - \phi_l}{c_h - c_l + \dfrac{b}{d}(\phi_l - c_l)} - (2-x_b) > 0. \tag{6.35}$$

7. Take-it-or-leave-it offer by the seller

It is beyond the scope of this paper to completely characterize the class of strategic bargaining protocols where the main result of the paper continues to hold. Instead, we choose a strategic bargaing model which is widely used in the literature to demonstrate that the formal result and intuition of the paper are not sensitive to the details of the trading rule.

7.1. **Bargaining protocol.**

We examine the bargaining protocol in which the seller makes an offer, and the buyer responds. For simplicity, we only consider the case in which the mass of the seller and the buyer is equal so that in the competitive market, the market should clear.

We consider the case where the mass of the buyers is two, the same as the sum of the mass of c_h and c_l type sellers of equal size: $x_b = 2$. As a result, in each period, there is an equal mass of buyers and sellers in the pool and a buyer and a seller in the pool are randomly matched with probability 1. Once they are matched, the seller makes an offer $p \in [0, \infty)$ to the buyer. The buyer then either accepts or rejects the offer. If the buyer accepts the offer, then they leave the pool and form a long term relationship. Otherwise, they both return to the pool, waiting for the next matching in the next period.

As in the main model, we calculate a stationary equilibrium in which the equilibrium strategy of each player depends only upon the status of the player: whether or not the player is in the long term relationship, and the distribution of the agents in the pool does not change over time.

7.2. **Analysis.**

We construct an equilibrium where c_h type sellers always offer p_h, c_l type sellers take a mixed strategy, offering p_h with probability $\lambda \in (0, 1)$ and p_l with $1 - \lambda$, and the buyers accept p_h with probability $\sigma \in (0, 1)$ and always accept p_l. We assume $p_h > p_l$.

Since the informed seller makes an offer, we can construct many stationary equilibria by manipulating beliefs off the equilibrium path. Inspired by Divinity (Banks and Sobel (1987)), we impose a simple restriction on belief off the equilibrum path: for any price higher than the equilibrium offer of the seller, the buyer believes that the fraction of c_h type sellers does not increase, and conditioned on the price lower than the equilibrium offer of c_h seller, the buyer assigns probability 1 to c_l type. We simply refer to this type of equilibrium as an equilibrium, whenever the meaning is clear from the context.

7.2.1. *c_h type seller.*

The value function of an c_h type seller in a long term relationship is essentially the same as in the main analysis, i.e.,

$$W_s^h(p_h) = \frac{(1-\beta)(p_h - c_h) + \beta(1-\delta)W_s^h}{1-\beta\delta}. \tag{7.36}$$

On the other hand, W_s^h is given by

$$W_s^h = \sigma W_s^h(p_h) + (1-\sigma)\beta W_s^h. \tag{7.37}$$

Substituting (7.36) into (7.37), we obtain, after some calculation,

$$W_s^h = \frac{\sigma}{1-(1-\sigma)\beta\delta}(p_h - c_h). \tag{7.38}$$

7.2.2. c_l *type seller.* Like in the main analysis, the value function of a c_l type seller in a long term relationship when agreed at price p ($p = p_h, p_l$) is given by

$$W_s^l(p) = \frac{(1-\beta)(p-c_l) + \beta(1-\delta)W_s^l}{1-\beta\delta}, \quad p = p_h, p_l. \tag{7.39}$$

On the other hand, W_s^l is given by

$$W_s^l = \lambda\sigma W_s^l(p_h) + \lambda(1-\sigma)\beta W_s^l + (1-\lambda)W_s^l(p_l). \tag{7.40}$$

Substituting (7.39) with $p = p_h, p_l$ into (7.40), we obtain

$$W_s^l = \frac{1}{1-(1-\sigma)\beta\delta}\left[\lambda\sigma(p_h - c_l) + (1-\lambda)(p_l - c_l)\right]. \tag{7.41}$$

7.2.3. *Buyer.* The value function of a buyer in a long term relationship when agreed at price p ($p = p_h, p_l$) is given by

$$W_b(p) = (1-\beta)(\phi(p) - p) + \beta\left[\delta W_b(p) + (1-\delta)W_b\right],$$

where

$$\phi(p_h) = \frac{\mu}{\mu + \lambda(1-\mu)}\phi_h + \frac{\lambda(1-\mu)}{\mu+\lambda(1-\mu)}\phi_l, \tag{7.42}$$

and

$$\phi(p_l) = \phi_l.$$

Calculating this, we obtain

$$W_b(p) = \frac{(1-\beta)(\phi(p) - p) + \beta(1-\delta)W_s^l}{1-\beta\delta}, \quad p = p_h, p_l. \tag{7.43}$$

On the other hand, W_b is given by

(7.44)

$$W_b = (\mu + \lambda(1-\mu))\,\sigma W_b(p_h) + (\mu+\lambda(1-\mu))\,(1-\sigma)\beta W_b + (1-\lambda)(1-\mu)W_b(p_l).$$

Substituting (7.43) with $p = p_h, p_l$ into (7.44), we obtain, after some calculation,

$$W_b = \frac{(\mu+\lambda(1-\mu))\,\sigma(\phi(p_h) - p_h) + (1-\lambda)(1-\mu)(\phi(p_l) - p_l)}{1-(\mu+\lambda(1-\mu))\,(1-\sigma)\beta\delta}. \tag{7.45}$$

7.2.4. *The balance equations.* The balance equations need to be modified, too. The balance equations for the two types of sellers are

$$\begin{aligned} \sigma z_s^h &= (1-\delta)(1-z_s^h) \\ (\sigma\lambda + (1-\lambda))z_s^l &= (1-\delta)(1-z_s^l). \end{aligned}$$

Therefore, we have

$$z_s^h = \frac{1-\delta}{1-\delta+\sigma} \tag{7.46}$$

$$z_s^l = \frac{1-\delta}{1-\delta+\lambda\sigma+(1-\lambda)}. \tag{7.47}$$

Thus, we obtain

$$\mu = \frac{z_s^h}{z_s^h + z_s^l} = \frac{\lambda\sigma + (1-\lambda) + (1-\delta)}{\lambda\sigma + (1-\lambda) + 2(1-\delta) + \sigma}. \tag{7.48}$$

7.2.5. *Indifference for a c_l seller.* Since c_l type sellers mix between p_h and p_l, they have to be indifferent between them, which gives us the following equation:

$$\sigma W_s^l(p_h) + (1-\sigma)\beta W_s^l = W_s^l(p_l). \tag{7.49}$$

7.2.6. *Indifference for a buyer.* Since the buyers take a mixed strategy between accepting and rejecting p_h, they have to be indifferent between them, which gives us the following equation:

$$W_b(p_h) = \beta W_b. \tag{7.50}$$

Similarly, the buyers should be indifferent between accepting and rejecting p_l; for if not, the L type sellers would increase the price from p_l. Therefore, we have

$$W_b(p_l) = \beta W_b. \tag{7.51}$$

7.3. **An equilibrium with involuntary unemployment and involuntary vacancy.** We construct an equilibrium by setting $p_h = c_h$ and $p_l = \phi_l$. We support this equilibrium with off-path beliefs where the buyer believes that the seller is an c_l type if $p < p_h$, and that the fraction of c_h type sellers for $p > p_h$ is the same as the one at $p = p_h$. This assumption implies $W_s^h = W_b = 0$ and $\phi(p_h) = c_h$. Then, from (7.48), we have

$$\phi(p_h) = \frac{\mu}{\mu + \lambda(1-\mu)}\phi_h + \frac{\lambda(1-\mu)}{\mu+\lambda(1-\mu)}\phi_l = c_h. \tag{7.52}$$

Substituting μ in (7.48) into (7.52), we obtain, after some calculation,

$$[(1-\delta) + (1-\lambda) + \lambda\sigma](\phi_h - c_h) = \lambda(1-\delta+\sigma)(c_h - \phi_l). \tag{7.53}$$

Therefore, we have

$$\lambda = \frac{(2-\delta)(\phi_h - c_h)}{(1-\sigma)(\phi_h - c_h) + (1-\delta+\sigma)(c_h - \phi_l)}. \tag{7.54}$$

This is strictly between 0 and 1 since all the coefficients are positive, and $c_h - \phi_l > \phi_h - c_h$ holds due to the condition of "lemon's problem is severe".

Next, substituting (7.39) with $p = p_h = c_h$ and $p = p_l = \phi_l$, and (7.45) into (7.49), we obtain

$$\begin{aligned}\Phi(\sigma, \Delta) \equiv (\phi_l - c_l) - 2\beta\delta(1-\sigma)(\phi_l - c_l) - \sigma[1 - \beta\delta(1-\sigma)](c_h - c_l) \\ - \beta\delta\lambda(1-\sigma)[\sigma(c_h - c_l) - (\phi_l - c_l)] = 0.\end{aligned} \tag{7.55}$$

Note that $\beta = e^{-b\Delta}$ and $\delta = e^{-d\Delta}$. This function Φ is continuous in σ. If $\sigma = 1$, then $\Phi = \phi_l - c_h < 0$. Therefore, we have a solution if $\Phi > 0$ at $\sigma = 0$. At $\sigma = 0$, we have

$$\Phi(0, \Delta) = (1 - 2\beta\delta) + \beta\delta \frac{(2-\delta)(\phi_h - c_h)}{(\phi_h - c_h) + (1-\delta)(c_h - \phi_l)}.$$

We would like to determine the sign of this expression in the limit as Δ converges to zero.

First, we multiply Φ by $(\phi_h - c_h) + (1-\delta)(c_h - \phi_l)$ and denote it by $\hat{\Phi}$. Since $\hat{\Phi}(0, \Delta)$ goes to zero in the limit, we take its derivative with respect to Δ and obtain

$$D \equiv \frac{\partial \hat{\Phi}(0,0)}{\partial \Delta} = (b + d)(\phi_h - c_h) + d(\phi_h + \phi_l - 2c_h).$$

The first term is positive, while the second term is negative due to the assumption that the lemon's problem is severe.[13]

If $D > 0$, then there exists $\sigma \in (0, 1)$ such that (7.55) holds. Hence, if $D > 0$, then there exists a pair (λ, σ) that satisfies the incentive conditions, and therefore, constitutes an equilibrium.

Repeating the same procedure, we obtain

$$S \equiv \frac{\partial \hat{\Phi}(0,0)}{\partial \sigma} = (\phi_l - c_l)(\phi_h + \phi_l - 2c_h) - (\phi_h - c_h)(c_h - \phi_l).$$

Therefore,

$$\left.\frac{d\sigma}{d\Delta}\right|_{\Delta=0} = -\frac{\partial \hat{\Phi}(0,0)/\partial \Delta}{\partial \hat{\Phi}(0,0)/\partial \sigma} = -\frac{D}{S}.$$

Since $D > 0$ holds, we must have $S < 0$. Both $D > 0$ and $S < 0$ hold if the lemon's problem is severe, but not too severe:

$$|\phi_h + \phi_l - 2c_h| < \phi_h - c_h.$$

Let us write $\sigma = k\Delta$ as a linear approximation where $k = -D/S$. Substituting this expression into the balance equation for c_h-type sellers, we have

$$k\Delta z_s^h = (1 - e^{-d\Delta})(1 - z_s^h).$$

---

[13] To be precise, we cannot define $\hat{\Phi}$ at $(0, 0)$. The expression here is the abbreviation of $\lim_{\Delta \to 0} \hat{\Phi}(0, \Delta)$.

Taking the limit of Δ going to zero, we obtain

$$\lim_{\Delta\to 0} z_s^h = \frac{d}{k+d},$$

which is strictly positive, i.e., we have involuntary unemployment and involuntary vacancy even in the limit of friction vanishing. Summarizing the above, we have the following theorem.

Theorem 7.1. *There exists an equilibrium in which the market fails to clear, i.e.,*

$$\lim_{\Delta\to 0} z_b z_s > 0.$$

The key factor for the result is $W_b = 0$. If this holds, then the buyer has no incentive to speed up the trade, which leads to delay in reaching agreement. Note that in the baseline model, market fails to clear if and only if $W_b = 0$ in the limit. The same is true in the free entry model. Indeed, in Moreno and Wooders (2010), $W_b = 0$ must hold to obtain delay. In this sense, all of these results have a common feature.

8. Concluding Remarks

This paper examines a dynamic matching model with adverse selection to see whether or not the market almost clears if search friction is small. We identify adverse selection as a fundamental source of the coexistence of unemployment and vacancy other than search friction and coordination failure caused by directed search.

One may be interested in what happens if, instead of commiting to a particular price level, the agents can commit to a certain price path during a long term relationship. If, like in some of the countries in Europe and Asia, workers are protected, then a certain pattern of wage structure such as a seniority wage system may emerge. The analysis of such cases remains for the future research.

The feature that is common in the equilibria that exhibit the coexistence in various models, including Moreno and Wooders (2010), our baseline model with a certain condition, our free entry model, and the model in Section 7 is that the buyers' value is driven down to zero, making them indifferent between trading now and in the future. We chose the random proposal model as a bargaining protocol mainly for the analytical convenience. The preliminary investigation reveals that the main conclusion of this paper is robust against the details of the bargaining protocol. Indeed, Section 7 demonstrated that the result is carried over to the model with a bargaining protocol in which the seller makes the ultimatum offer in each period to the buyer. Whether this result holds in a wider variety of models or not remains the task for the future research.

Appendix A. Proof of Proposition 4.1

Define $O(\Delta)$ as a function that vanishes at the rate of Δ:

$$\lim_{\Delta \to 0} \frac{O(\Delta)}{\Delta} < \infty.$$

Lemma A.1. $\lim_{\Delta \to 0} (\pi_p)^2 \leq O(\Delta)$

Proof. The second term of the buyer's value function and $W_b < \infty$ imply the statement. □

Lemma A.2. $\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} < \infty$.

Proof. Suppose $\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} = \infty$. Since $\lim_{\Delta \to 0} W_s^l < \infty$,

$$\frac{\zeta \pi_p}{1-\beta\delta} \mathsf{E}(p - W_s^l - c_l \mid \Pi_p) < \infty.$$

Under the hypothesis of the proof,

$$\lim_{\Delta \to 0} \mathsf{E}(p - W_s^l - c_l \mid \Pi_p) = 0.$$

Since $\pi_p > 0$ and $\lim_{\Delta \to 0} \pi_p = 0$,

$$0 < \phi^e - W_b - c_l - W_s^l \to 0.$$

Recall

$$\phi_l < c_h.$$

Thus,

$$\phi_l - W_b < c_h + W_s^h$$

and the gap between the left and the right hand sides does not vanish as $\Delta \to 0$. Since $\pi_s > 0$,

$$c_l + W_s^l < \phi_l - W_b < c_h + W_s^h < \phi^e - W_b$$

while

$$\phi^e - W_b - c_l - W_s^l \to 0.$$

This is a contradiction. □

Based upon these two observations, we conclude that the high quality seller's equilibrium payoff vanishes as $\Delta \to 0$, which proves the first part of Proposition 4.1.

Lemma A.3. $\lim_{\Delta \to 0} W_s^h = 0$.

Proof. Apply Lemmata A.1 and A.2 to W_s^h. □

Since $\pi_s > 0$, an c_l seller and a buyer trades with a positive probability, which imposes an upper bound on $W_s^l + W_b$.

Lemma A.4. $W_s^l + W_b < \phi_l - c_l$.

Proof. A direct implication of $\pi_s > 0$. □

The next lemma shows that the low quality seller cannot be completely sorted out in a semi-pooling equilibrium, even in the limit as $\Delta \to 0$. As the pool contains a non-negligible portion of low quality sellers, the buyer needs to sort out the sellers, which is costly for the buyer and for the society as a whole, even if the friction vanishes. On the other hand, the low quality seller has an option to imitate the high quality seller, which provides significant bargaining power to a low quality seller when she is matched to a buyer.

Lemma A.5. $\lim_{\Delta \to 0} \mu_l > 0$.

Proof. Suppose $\lim_{\Delta\to 0}\mu_l = 0$. Then $\lim_{\Delta\to 0}\phi(\mu_l) = \phi_h$ holds. Thus, from (3.18), Lemmata A.3 and A.4 together with $W_s^l \geq 0$, we have

$$\lim_{\Delta\to 0}\pi_p = \lim_{\Delta\to 0} C[\phi_h - c_h - W_b - W_s^h] \geq C[(\phi_h - c_h) - (\phi_l - c_l)] > 0,$$

which contradicts with Lemma A.1. □

As in Lemma A.2, we can compute the rate at which $\zeta\pi_s$ vanishes.

Lemma A.6. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} < \infty$.

Proof. Suppose $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = \infty$. Then from Lemma A.2 and the balance equations of the sellers, $\lim_{\Delta\to 0}\mu_l = 0$ holds, which contradicts to Lemma A.5. □

The next lemma is the seller's counterpart of Lemma A.1.

Lemma A.7. $\lim_{\Delta\to 0}\pi_s \leq O(\Delta)$.

Proof. This statement is directly implied by Lemma A.5 and (3.16). □

A corollary of Lemma A.7 is that the sum of the long run average payoffs of a buyer and c_l seller converges to $\phi_l - c_l$, which proves the second part of Proposition 4.1.

Lemma A.8. $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - c_l$.

Proof. From Lemma A.7 together with (3.17), we have

$$\lim_{\Delta\to 0}\pi_s = \lim_{\Delta\to 0} C[(\phi_l - c_l) - (W_b + W_s^l)] = 0.$$

□

Appendix B. Proof of Lemma 4.2

From (3.19), (3.20) and (3.21), we know that in order to investigate the asymptotic properties of z_b and z_s, we need to understand the asymptotic properties of $\zeta\pi_p/(1-\delta)$ and $\zeta\pi_s/(1-\delta)$.

Lemma B.1. $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} > 0$

Proof. Suppose that $\lim_{\Delta\to 0}\frac{\zeta\pi_s}{1-\beta\delta} = 0$. From the balance equations of the sellers, we have

$$\frac{\mu_l}{1-\mu_l} = \frac{\frac{\pi_p\zeta}{1-\delta} + 1}{\frac{\pi_s\zeta}{1-\delta} + \frac{\pi_p\zeta}{1-\delta} + 1} \to 1$$

which implies that

$$\mu_l \to \frac{1}{2}.$$

Since the lemons problem is severe (assumption $A3$),

$$\phi(\mu_l) - c_h \to \frac{\phi_h + \phi_l}{2} - c_h < 0.$$

Recall that $W_s^h \to 0$. Since any equilibrium must be semi-pooling, $\pi_p > 0$. For a sufficiently small $\Delta > 0$, however,

$$0 < \phi(\mu_l) - W_b - W_s^h - c_h \leq \phi(\mu_l) - c_h \to \frac{\phi_h + \phi_l}{2} - c_h < 0$$

which is impossible. □

Lemma B.2.
$$0 < \lim_{\Delta \to 0} \frac{\pi_s}{\pi_p} < \infty.$$

Proof. Since we have
$$0 < \lim_{\Delta \to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$
by way of Lemmata A.6 and B.1 , and
$$\lim_{\Delta \to 0} \frac{\pi_p \zeta}{1-\delta} < \infty,$$
by way of Lemma A.2,
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} < \infty.$$
holds. To prove
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} > 0$$
by way of contradiction, suppose that
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} = 0.$$
Since
$$0 < \lim_{\Delta \to 0} \frac{\pi_s \zeta}{1-\delta} < \infty,$$
$$\lim_{\Delta \to 0} \frac{\pi_p}{\pi_s} = 0$$
implies
$$\lim_{\Delta \to 0} \frac{\pi_p \zeta}{1-\delta} = 0.$$
We claim that $\zeta \to 0$ as $\Delta \to 0$ under the hypothesis of the proof. If
$$\lim_{\Delta \to 0} \zeta > 0,$$
then $\pi_s = O(\Delta)$ and $\pi_p = O(\Delta)$. As a result,
$$\lim_{\Delta \to 0} W_s^l = \lim_{\Delta \to 0} W_b = 0,$$
which is impossible since
$$W_b + W_s^l \to \phi_l - c_l.$$
□

Lemma B.3. $\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} > 0$

Proof. Note
$$\lim_{\Delta \to 0} \frac{\zeta \pi_p}{1-\beta\delta} = \lim_{\Delta \to 0} \frac{\zeta \pi_s}{1-\beta\delta} \frac{\pi_p}{\pi_s}.$$
The desired conclusion follows from Lemmata B.1 and B.2. □

Lemma B.4. $\lim_{\Delta \to 0} \mathsf{E}[p|\Pi_p] = c_h$.

Proof. Since $\lim_{\Delta \to 0} \pi_p = 0$, $\Pi_p = [c_h + W_s^h, \phi^e(p) - W_b]$ shrinks to a single point. Since $\lim_{\Delta \to 0} W_s^h = 0$, all points in Π_p converge to c_h, from which the conclusion follows. □

Lemma B.5. $\lim_{\Delta \to 0} W_s^l > 0$

Proof. Recall the equilibrium value function of W_s^l, and observe that the second term of the value function is strictly positive, even in the limit as $\Delta \to 0$. □

We are ready to prove Lemma 4.2. Note

$$\frac{W_s^l}{W_b} = \frac{A\zeta\pi_s^2 + \zeta\pi_p \mathsf{E}[p - c_l - W_s^l|\Pi_p]}{A\left(\mu_l\pi_s^2 + \pi_p^2\right)}.$$

Thus,

$$\frac{\mu_l W_s^l}{\zeta W_b} \propto \frac{\mu_l\zeta\pi_s^2 + \mu_l\zeta\pi_p \mathsf{E}(p - c_l - W_s^l|\Pi_p)}{\mu_l\zeta\pi_s^2 + \zeta\pi_p^2} = \frac{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \mu_l \mathsf{E}(p - c_l - W_s^l|\Pi_p)}{\mu_l\pi_s\frac{\pi_s}{\pi_p} + \pi_p}. \tag{B.56}$$

The denominator converges to zero by way of Lemmata A.1, A.7, and B.2, while the numerator converges to a value greater than or equal to $\mu_l(c_h - \phi_l) > 0$ due to Lemma B.4 and $\lim_{\Delta\to 0} W_s^l \leq \phi_l - c_l$. Therefore, since $\lim_{\Delta\to 0} \mu_l W_s^l > 0$, $\zeta W_b \to 0$.

Appendix C. Proof of Lemma 4.4

Suppose $\lim_{\Delta\to 0} z_b > 0$. Then Lemma 4.2 implies $\lim_{\Delta\to 0} W_b = 0$, which in turn implies $\lim_{\Delta\to 0} W_s = \phi_l - c_l$ due to Proposition 4.1. We derive (4.24) from W_s^l by using the fact that the first term converges to zero, and Lemma B.4. As for (4.25), note that $\mu_l = z_s^l/z_s$. Taking the limit of this expression and equating it with $\lim_{\Delta\to 0} \mu_l = \frac{\phi_h - c_h}{\phi_h - \phi_l}$, we derive (4.25).

Appendix D. Proof of Theorem 4.5

We prove the sufficiency of (4.28) in multiple steps.

Proposition D.1. *Suppose that* $\lim_{\Delta\to 0} z_b = 0$. *Then,*

(1) $\lim_{\Delta\to 0} z_s = 2 - x_b$.
(2) $\frac{\pi_p}{1-\delta} \to \infty$ *and* $\frac{\pi_s}{1-\delta} \to \infty$ *as* $\Delta \to 0$.
(3) $\lim_{\Delta\to 0} W_b \geq 0$ *and the equality holds only if* (4.28) *is violated with equality.*
(4) (4.28) *is violated.*

Proof. Suppose $\lim_{\Delta\to 0} z_b = 0$.

(1) follows from the fact that $2 - z_s = x_b - z_b$.

(2) Note that $\zeta \to 0$ if and only if $z_b \to 0$. Lemma B.1 and Lemma B.3 imply that $\frac{\pi_p}{1-\delta} \to \infty$ and $\frac{\pi_s}{1-\delta} \to \infty$ as $\Delta \to 0$.

(3) To simplify notation, let us write

$$\begin{aligned}
\bar{\mu} &= \lim_{\Delta\to 0} \mu_l = \frac{\phi_h - c_h}{\phi_h - \phi_l} \\
\bar{Q}_s &= \lim_{\Delta\to 0} \frac{\zeta\pi_s}{1-\delta} \\
\bar{Q}_p &= \lim_{\Delta\to 0} \frac{\zeta\pi_p}{1-\delta}.
\end{aligned}$$

Under the assumption that $\zeta \to 0$, one can derive from the balance equations that

$$\frac{x_b}{2 - x_b} = \bar{\mu}\bar{Q}_s + \bar{Q}_p$$

and

$$\frac{\bar{\mu}}{1-\bar{\mu}} = \frac{\bar{Q}_p + 1}{\bar{Q}_s + \bar{Q}_p + 1}.$$

From the value function of c_l seller, one can show that

$$\lim_{\Delta\to 0} W_s^l = \frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p}.$$

Since

$$W_s^l + W_b \to \phi_l - c_l,$$

$\lim_{\Delta\to 0} W_b > 0$ if and only if

$$\frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p} < \phi_l - c_l.$$

We know that if $\bar{Q}_p = Q_p$, then

$$\frac{\frac{d}{b+d}\bar{Q}_p(c_h - c_l)}{1 + \frac{d}{b+d}\bar{Q}_p} = \phi_l - c_l.$$

Thus, $\lim_{\Delta\to 0} W_b > 0$ if and only if $\bar{Q}_p < Q_p$. One can show that $\bar{Q}_p$ solves

$$\bar{Q}_p + 1 = \left(1 + \frac{d}{d+b}\bar{Q}_p\right)\left(\frac{\phi_h - \phi_l}{c_h - c_l}\frac{1}{2 - x_b}\right),$$

where we use the balance equations, $\lim_{\Delta\to 0} W_s^l + W_b = \phi_l - c_l$, and

$$\bar{\mu}\phi_l + (1-\bar{\mu})\phi_h = c_h + \lim_{\Delta\to 0} W_b.$$

Note that $\bar{Q}_p \le Q_p$ if and only if (4.28) is violated, and the equality holds only if (4.28) is violated with an equality.

(4) follows from the last part of the proof of (3). □

References

Akerlof, G. A. (1970): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84(3), 488–500.

Azariadis, C. (1975): "Implicit Contracts and Underemployment Equilibria," *Journal of Political Economy*, 83(6), 1183–1202.

Banks, J. S., and J. Sobel (1987): "Equilibrium Selection in Signaling Games," *Econometrica*, 55(3), 647–661.

Blouin, M. R., and R. Serrano (2001): "A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies*, 68, 323–346.

Burdett, K., S. Shi, and R. Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), 1060–1085.

Burdett, K., and R. Wright (1998): "Two-Sided Search with Nontrasferable Utility," *Review of Economic Dyanmics*, 1(1), 220–245.

Chang, B. (2012): "Adverse Selection and Liquidity Distortion in Decentrailed Markets," Ph.D. thesis, Northwestern University.

Cho, I.-K., and A. Matsui (2013): "A Search Theoretic Foundation of Nash Bargaining Solution," *Journal of Economic Theory*, 148, 1659–1688.

——— (2017): "Foundation of competitive equilibrium with non-transferable utility," *Journal of Economic Theory*.

Diamond, P. A. (1971): "A Model of Price Adjustment," *Journal of Economic Theory*, 3(2), 156–168.

Guerrieri, V., R. Shimer, and R. Wright (2010): "Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78(6), 1823–1862.

Gul, F., and A. Postlewaite (1992): "Asymptotic Efficiency in Large Exchange Economies with Asymmetric Information," *Econometrica*, 60(6), 1273–1292.

Keynes, J. M. (1936): *The General Theory of Employment, Interest and Money*. Macmillan.

Lagos, R. (2000): "An Alternative Approach to Search Frictions," *Journal of Political Economy*, 108, 851–873.

Matsui, A., and T. Shimizu (2005): "A Theory of Money and Market Places," *International Economic Review*, 46.

Moreno, D., and J. Wooders (2010): "Decentralized Trade Mitigates the Lemons Problem," *International Economic Review*, 51(2), 383–399.

Mortensen, D. T., and C. A. Pissarides (1994): "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies*, 208, 397–415.

Rubinstein, A., and A. Wolinsky (1985): "Equilibrium in a Market with Sequential Bargaining," *Econometrica*, 53, 1133–1150.

Shapiro, C., and J. E. Stiglitz (1984): "Equilibrium Unemployment as a Worker Discipline Device," *American Economic Review*, 74(3), 433–444.

Stiglitz, J. E., and A. Weiss (1981): "Credit Rationing in Markets with Imperfect Information," *American Economic Review*, 71(3), 393–410.

Wolinsky, A. (1990): "Information Revelation in a Market with Pairwise Meetings," *Econometrica*, 58(1), 1–23.

Department of Economics, University of Illinois, 1407 W. Gregory Drive, Urbana, IL 61801 USA, Hanyang University and Federal Reserve Bank of St. Louis
E-mail address: inkoocho@illinois.edu
URL: https://sites.google.com/site/inkoocho

Faculty of Economics, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
E-mail address: amatsui@e.u-tokyo.ac.jp
URL: http://www.e.u-tokyo.ac.jp/~amatsui