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A TWO-STAGE MODEL OF ASSIGNMENT AND MARKET

AKIHIKO MATSUI AND MEGUMI MURAKAMI

Abstract. This paper studies a two-stage economy where the non-monetary assignments of indivisible objects are followed by market transactions. In this economy, there are finitely many players and finitely many types of indivisible objects and one divisible good called money. Every player demands at most one object besides money. The first stage is governed by a non-monetary assignment mechanism, while the second stage is governed by the market. As a mechanism in the first stage, this paper considers the Boston mechanism and the deferred acceptance algorithm. This paper defines perfect market equilibrium (PME) where the second stage outcome is a market equilibrium both on and off the equilibrium paths, and the first stage strategy profile is a Nash equilibrium of the mechanism, taking the second stage outcomes as given. This paper then analyzes two situations, the economies with and without money. We also applied our analysis to the college admission problem: some players (firms) cannot obtain objects (degrees) in the first stage, waiting for some other players (students) obtaining them, and buy the objects (hire the students with degrees) through the market in the second stage. This paper provides us with some necessary and sufficient conditions under which efficiency and stability are guaranteed.

Keywords: two-stage economy, assignment mechanism, market, indivisible object, perfect market equilibrium, scarcity, priority cycle, stability

JEL Classification Numbers: C78, D41, D47, D51

1. Introduction

This paper studies a two-stage model where the non-monetary assignments of indivisible objects are followed by market transactions. This model is related to the following couple of situations. First, consider a problem of college admission where students select a college to be admitted. They do so strategically, taking into account their future job prospects, rather than truthfully expressing their intrinsic preferences such as their love for campus. Next, consider the following historical case of the United States: after Homestead Act was enacted in 1862, pioneers of the great prairie obtained a piece of land (160 acres) in return for living there and cultivating the land. After acquisition, the lands were freely traded in the market. Also, imagine several firms that strategically develop new technology to obtain patents. The intellectual property rights are assigned to firms on the first-come-first-served basis. After the acquisition of the rights, they can sell

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their patents to other firms or keep them and commercialize the invented technology in the market. The fourth example is a situation in which office spaces in a newly constructed building are alloted to faculty members. In the first stage, the department assigns rooms to the members based on some predetermined rule. Then, the members are allowed to exchange their rooms afterwards.

A common observation in the above cases is that there are two stages, an assignment stage and a market stage, and that assignment through a formal or informal non-price mechanism of the first stage affects and is affected by what people obtain in the subsequent market, and therefore, players therein would choose objects to obtain in the assignment stage in a strategic manner, taking into account their prospects in the market stage. This observation raises a number of questions: under what conditions does equilibrium exist? when does the first stage assignment matter in terms of efficiency? what do we miss if the assignment stage and the market stage are separately analyzed?

In order to examine these questions, we construct a two-stage model. In this model, there are finitely many players and finitely many types of indivisible objects. In addition, one divisible good called money may or may not be available. Every player demands at most one object besides money. The players may face different priorities at each object type in the first stage. Each object has a limited amount of capacity, called quota. Each player has a quasi-linear utility function.

The first stage is governed by a non-monetary assignment mechanism. A mechanism is a pair of the set of strategy profiles and an assignment rule. Given a mechanism, players simultaneously choose strategies to obtain one unit of some object. Then, the mechanism chooses an allocation based on the selected strategy profile and priorities. That is, if the number of the players who choose a certain object type exceeds its quota, then the players with top priority will obtain the objects up to the quota; otherwise, the objects are alloted to all the players who choose it. Throughout this paper, we assume that the mechanism is either the Boston mechanism or the deferred acceptance algorithm.

The second stage is governed by the market. The players are endowed with the objects assigned in the first stage as well as money. The priority no longer matters in the second stage. The players can trade objects as a price taker. Each player's payoff is determined by the indivisible good and the money held at the end of the second stage. In particular, what they obtain in the first stage matter only to the extent that it affects the final allocation in the second stage.

We introduce an equilibrium concept called perfect market equilibrium (PME) to analyze these situations. PME requires that a market equilibrium be realized in each market of the second stage, and that each player selects an optimal strategy in the mechanism, taking into consideration what will happen in the second stage. In order to capture players' incentives in the first stage, we define an induced game, where the payoff of each strategy profile is defined by the corresponding market equilibrium outcome. PME is a Nash equilibrium in this induced game.

When money is available in the economy, a refined concept of PME, Permutation independent PME (PIPME), is considered. It requires that the price profiles of the second stage should be the same between two initial endowment profiles as long as their total endowments are the same[1]. This concept captures the idea of anonymity and (partial) price-taking behavior. PIPME reflects, in addition to perfection, the idea of anonymity in the sense that changes in object holders would not change the price system. It also reflects the idea of price taking behavior in the sense that even if one changes his/her strategy in the assignment stage, it would not affect the second stage price system.

[1]PIPME is considered only when money is available since it restricts the price system.

In the analysis, two criteria are used to evaluate the allocation in PME: one is Pareto optimality, and the other is efficiency. If an allocation is Pareto optimal, then there is no allocation where all the players weakly prefer and at least one player strictly prefers to this allocation. An efficient allocation maximizes the social welfare, which is equal to the sum of the players' utility values. We also introduce ω-optimality and ω-efficiency given an endowment ω of the second stage, which correspond to optimality and efficiency, respectively. Given ω, an ω-optimal (resp. ω-efficient) allocation is not necessarily an optimal (resp. efficient) allocation, especially when there are some unassigned objects in the first stage.

With this two-stage model and solution concepts, we analyze two types of situations. The first type of situation, analyzed in Section 3 is the one in which players have money. This situation corresponds to an assignment stage followed by monetary transactions.

The existence of market equilibrium in the second stage is guaranteed as shown by Quinzii (1984). Therefore, PME always exists in this case since the first stage outcome of PME is simply a Nash equilibrium given the second stage outcomes. We show that PME object allocation is unique and efficient under any utility value if and only if objects are scarce. The proof of sufficiency is an application of the first fundamental theorem of welfare economics.[2] The assumption of scarcity, however, is essential to the results. For the proof of necessity, we construct an economy where efficiency and uniqueness do not hold due to the lack of scarcity. If the scarcity assumption is violated, an $\omega-$ efficient allocation in the market may not coincide with the unique efficient allocation in the two-stage economy.

An application to a college admission problem is analyzed in Section 4. It is the one in which the buyers and the sellers of the objects in the second stage market are inherently separated: the players who obtain indivisible objects in the first stage will turn to be the sellers of the objects in the subsequent market. Interpretation is that students choose a college to be admitted, while firms hire the studetns based on the colleges they graduate from. The students strategically choose a college, taking into account their future job prospects. We show that PME always exists as market equilibrium exists under any value profiles if and only if the objects are scarce for both students and firms.

The second type of situation, analyzed in Section 5, is an assignment stage followed by exchanges where there is no money, or monetary transactions are considered inappropriate.

The existence of market equilibrium in the second stage is not guaranteed without various conditions. For example, if the quota of some object exceeds one, market equilibrium may not exist. This also implies that the existence of PME is not guaranteed unless the quota of each object type is limited to one. On the other hand, Pareto optimality of PME does not require even scarcity on condition that it exists. Again, the proof for optimality is an application of that of the first fundamental theorem of welfare economics.

To examine the relationships between the perfection of players and the first stage mechanism, we relate PME to stability. An allocation is said to be stable if every player prefers his/her assignment to any object that is held by another player whose priority is lower than the player in question and to any unassigned object. We introduce another equilibrium concept, called stable market equilibrium (SME) for the analysis without money. It imposes stability on the market equilibrium object allocation. SME, unlike PME, considers neither the incentive to deviate in the first stage nor off-the-equilibrium outcomes, and therefore, it is much easier to construct SME than

[2]See, e.g., Mas-Colell, Whinston, Green, et al. (1995).

PME. We show that the condition that there is no priority cycle is equivalent to the conditions that SME exists, and there exists a PME of which object allocation is the same as that of SME.

Since we analyze a two-stage model that consists of non-monetary assignments in the first stage and market transactions in the second, our analysis is based on a variety of existing literature even if we limit our attention to the papers that are directly related to the present one.

The present model closely follows the literature on assignment problems. The college admissions problem is adopted from Gale and Shapley (1962) and Roth and Sotomayor (1989). Sotomayor (2008) formulates a game form and define a Nash equilibrium to analyze stable matching mechanisms.

Ergin (2002) shows that no cycle of priority, or acyclicity, is equivalent to Pareto optimality of the outcome in the deferred acceptance mechanism. This condition of acyclicity turns out to play a critical role in relating stable allocations to PME allocations.[3]

This paper is based upon some results in the existing literature on markets with indivisible goods. Shapley and Scarf (1974) shows non-emptiness of core and existence of competitive equilibrium when there is no money. We use their result directly in proving the existence of market equilibrium in the case of no money. Kaneko (1982) shows non-emptiness of core under no-transferable utility. Wako (1984) shows strong core is inside the set of competitive equilibrium and conditions under which strong core exists. Quinzii (1984) shows the existence of competitive equilibrium in an economy with indivisible goods and money. We use this result directly in stating the existence result of market equilibrium in the case with money.

If we view the second stage endowment as the assignment of property rights, then the analysis of the present paper is related to Coase's theorem (see Coase (1960)). In the present context, the theorem implies that irrespective of the assignment of property right, the market will lead to an efficient allocation. Papers related to Coase's theorem are abundant. In the present context, it is worth mentioning Demsetz (1964), which states that under smooth markets, zero pricing of scarce good does not lead to inefficiency, and Jehiel and Moldovanu (1999), which considers assignment with resale and shows that the assignment of property right is irrelevant if there are resale processes. In the present paper, if monetary transaction is possible and the scarcity condition holds, then the situation becomes a special case of Jehiel and Moldovanu (1999). In other cases, however, their presumption does not hold, and the result may not hold in general.

The literature on mechanisms with renegotiation is also related. Maskin and Moore (1999) considers a two-stage model where a mechanism is implemented in the first stage, but the players cannot commit to its outcome in the second stage and may renegotiate to move to a Pareto-improving outcome. The present paper studies a more specific model to obtain more specific results than theirs.

The rest of the paper is organized as follows. Section 2 presents a model and solution concepts as well as some preliminary results. Section 3 studies situations with money. Section 4 studies a situation where the population is divided into two groups, students and firms. Section 5 studies situations without money. Some proofs and the definitions of some mechanisms are relegated to appendices.

[3] See also Kojima and Manea (2010), which takes axiomatic approaches on deferred acceptance mechanisms.

2. Model

We consider a two-stage economy. In the first stage, players play a game to obtain objects, while in the second stage, the market opens to allocate objects and money, if any. The object allocation in the first stage is governed by a mechanism. On the other hand, an object allocation in the second stage is determined through a pure exchange economy based on the profile of the initial endowments, objects and money. In this model, the initial object endowment profile in the second stage is the outcome of the first stage.

2.1. **Preliminaries.** N is a finite set of players. O is a finite set of objects. Assume $|N| \geq 2$ and $|O| \geq 2$. There is a null object, denoted ϕ. We may call ϕ an object and any a in O a tangible object whenever convenient. Let $\bar{O} = O \cup \{\phi\}$. The objects are indivisible, and each agent demands at most one unit of the object. For any $a \in O$, a has a quota $q^a \in \{1, 2, \ldots\}$. Also, let q^ϕ be a pseudo quota for ϕ. We assume $q^a < |N|$ for $a \in O$, while $q^\phi = |N|$. A quota profile is denoted by $q = (q^a)_{a \in O}$. Given a vector $\mu = (\mu_i)_{i \in N} \in \bar{O}^N$ and $a \in \bar{O}$, let $\mu^a = \{i \in N | \mu_i = a\}$ be the set of the players who hold a. An object allocation is μ that satisfies $|\mu^a| \leq q^a$ for all $a \in O$. $A^+ = \{\mu \in \bar{O}^N \mid \forall a \in \bar{O}\ |\mu^a| \leq q^a\}$ is the set of all the object allocations.

We consider two classes of economies, one with money and the other without. If there is money, an *allocation* is given by $x = (\mu, m) \in X \equiv A^+ \times \mathbb{R}^N$ with $\sum_{i \in N} m_i \leq 0$. If there is no money, an *allocation* is given by $(\mu, 0)$ or simply μ.

We consider quasi-linear utility functions, i.e., for every $i \in N$, the utility function $u_i : \bar{O} \times \mathbb{R} \to \mathbb{R}$ of agent i is given by

$$u_i(a_i, m_i) = v_i(a_i) + m_i.$$

In case of no money, we interchangeably write $u_i(a_i) = u_i(a_i, 0) = v_i(a_i)$. We assume that $v_i(a) \neq v_i(b)$ holds for all $a \neq b$. Moreover, we assume that these values are generic unless otherwise mentioned. In particular, we assume that for all $N', N'' \subset N$, and all allocations $\mu', \mu'' \in A^+$ with $\mu' \neq \mu''$,

$$\sum_{i \in N'} v_i(\mu'_i) \neq \sum_{j \in N''} v_j(\mu''_j) \tag{2.1}$$

holds.

We use two criteria to evaluate allocations in terms of utility. One is Pareto criterion and the other is social welfare. Consider two allocations x and x' in X. We say that x *Pareto dominates* x' if for all $i \in N$, $u_i(x_i) \geq u_i(x'_i)$ holds, and for some $j \in N$, $u_j(x_j) > u_j(x'_j)$ holds. An allocation x is *Pareto optimal* if there is no allocation that Pareto dominates x.

The second criterion is social welfare. For each allocation (μ, m), a *social welfare* is given by $W(\mu) = \sum_{i \in N} v_i(\mu_i)$. We say that (μ, m) is *efficient* if $\mu \in \arg\max_{\mu' \in A^+} W(\mu')$ holds.

2.2. **The first stage: Assignment.** In the first stage, the players obtain objects based on priority through a mechanism. As the first stage mechanism, we consider the Boston mechanism (Boston) and the deferred acceptance algorithm (DA).

A mechanism is given by a pair

$$M = \langle \Sigma, \lambda \rangle,$$

where $\Sigma = (\Sigma_i)_{i \in N}$ is the set of strategy profiles with Σ_i being the set of i's strategies and $\lambda : \Sigma \to A$ is an outcome function where $A = \times_{i \in N} A_i$, and we assume the set A_i of the available objects for $i \in N$ is either $\bar{O}$ or $\{\phi\}$.

For every $a \in O$, $\succ_a$ is a strict total order at $a \in O$ over the set of the players whose available objects are $\bar{O}$. It satisfies transitivity, asymmetry, and has no non-comparable pairs[4]. It defines the order of players' priority at object a, i.e., $i \succ_a j$ means that i has higher priority than j at a. Let $\succ = (\succ_a)_{a \in O}$ be the priority profile at all the objects.

The Boston mechanism works as follows.

(Boston)

Each player submits a list of objects ordered from the top to the bottom, i.e., for each $i \in N$ with $A_i = \bar{O}$, player i's list is given by $\left(a_i^1, \ldots, a_i^{|\bar{O}|}\right)$, and for each $i \in N$ with $A_i = \{\phi\}$, player i's list is simply (ϕ).

The rest is determined by the algorithm:

Step 1: Start with the top $(a_i^1)_{i \in N}$ of the players' respective lists.

$\star$ For each $a \in \bar{O}$, if the number of the players choosing a does not exceed q^a, i.e., $|\{i \in N | \ a_i^1 = a\}| \leq q^a$, then they are *irreversibly* assigned to a.

$\star$ If the number of the players choosing a exceeds q^a, then the top q^a players in terms of priority are *irreversibly* assigned to a, and the rest go to the next step with the second objects in their respective lists.

Step t $(t > 1)$**:** Repeat $\star$'s in Step 1 with remaining objects and remaining players. If the object a_i^t chosen by player i has been assigned up to its quota, or if the remaining units of this object are taken by the other players in this step, then the player goes to the next step with the $(t+1)$th object a_i^{t+1} in the list.

Terminate the process when all the players are assigned to an object in $\bar{O}$.

The deferred acceptance algorithm works as follows.

(DA)

Each player submits a list of objects ordered from the top to the bottom, i.e., for each $i \in N$ with $A_i = \bar{O}$, player i's list is given by $\left(a_i^1, \ldots, a_i^{|\bar{O}|}\right)$, and for each $i \in N$ with $A_i = \{\phi\}$, player i's list is simply (ϕ).

The rest is determined by the algorithm.

Step 1: Start with $(a_i^1)_{i \in N}$, the first object of the players' respective lists.

$\star$ For each $a \in \bar{O}$, if the number of the players choosing a does not exceed q^a, then they are *temporarily* assigned to a.

$\star$ If the number of the players choosing a exceeds q^a, then the top q^a players in terms of priority are *temporarily* assigned to a, and the rest go to the next step with the second objects in their respective lists.

Step t $(t > 1)$**:** Those assigned to a before and those who choose a in this step compete for a, and repeat $\star$'s in Step 1 where we replace the second objects with $(t+1)$th objects.

Terminate the process when all the players are assigned to an object in $\bar{O}$.

Note that in the both mechanisms, player i with $A_i = \{\phi\}$ always obtains ϕ in the first step of the algorithm.

[4]A binary relation $\succ_a$ over N is said to have *no non-comparable pairs* if $i \neq j$ implies either $i \succ_a j$ or $j \succ_a i$.

2.3. **The second stage: Market.** The players participate in the market in the second stage. To begin with, several related concepts are defined given the initial object allocation ω, which is the outcome of the first stage. Given $\omega \in A$ and $a \in O$, let $|\omega^a|$ be the *total endowment* of object a in the second stage. We write a total endowment profile, or simply total endowment, $|\omega| = (|\omega^a|)_{a \in O}$. Given an initial object allocation $\omega \in A$ of the second stage, an allocation $x = (\mu, m) \in X$ is *ω-feasible* if for all $a \in O$, $|\mu^a| \leq |\omega^a|$ holds. A^ω denotes the set of ω-feasible allocations. Also, $O^\omega = \{a \in O \| |\omega^a| > 0\}$ is the set of feasible objects, and $\bar{O}^\omega = O^\omega \cup \{\phi\}$.

Note that the quantity restriction is only on the objects in O, i.e., not on ϕ. Next, ω-Pareto optimality and ω-efficiency are defined.

Definition 2.1. *Given an initial object allocation ω of the second stage, an allocation x is* ω-Pareto optimal (ω-optimal) *if there does not exist an ω-feasible allocation x' that Pareto dominates x. Also, an allocation (μ, m), or simply μ, is* ω-efficient *if there does not exist an ω-feasible allocation (μ', m') such that $W(\mu') > W(\mu)$.*

The following lemma states the relationship between ω-optimality (resp. ω-efficiency) and Pareto optimality (resp. efficiency). It is a direct consequence of the respective definitions.

Lemma 2.1. *If $|\omega| = q$ holds, then an ω-optimal (resp. ω-efficient) allocation is also Pareto optimal (resp. efficient).*

Proof.
Suppose that $|\omega| = q$ holds. Then $A^\omega = A^+$ holds. Thus, the definition of ω-optimality (resp. ω-efficiency) becomes identical to that of Pareto optimality (resp. efficiency). □

The concept we use for the second stage is market equilibrium.

Definition 2.2. *Given $\omega \in A$, $(p, x) = (p, \mu, m) \in \mathbb{R}_+^{\bar{O}^\omega} \times A^\omega \times \mathbb{R}^N$ is a* market equilibrium *with money under ω if $p_\phi = 0$ holds, and*

(1) $\forall i \in N \; p_{\mu_i} + m_i = p_{\omega_i}$,
(2) $\forall i \in N \forall a \in \bar{O} \; [v_i(\mu_i) + m_i \geq v_i(a) + p_{\omega_i} - p_a)]$,
(3) $\forall a \in O^\omega \; [|\mu^a| \leq |\omega^a|] \wedge [|\mu^a| < |\omega^a| \Rightarrow p_a = 0]$.

In case of no money, we have $m = 0$.

Note that Definition 2.2, especially $p_\phi = 0$ and (3), together with ω-feasibility implies that the objects in O are free disposal.

2.4. **The two-stage economy and perfect market equilbrium.** We combine the two stages. First, we introduce an induced game.

Definition 2.3. *Given a mechanism $M = \langle \Sigma, \lambda \rangle$ and a profile $(p(\omega), x(\omega))_{\omega \in A}$, player i's* induced payoff *is given by*

$$\tilde{u}_i(\sigma) = u_i(x(\lambda(\sigma))).$$

Given a mechanism $M = \langle \Sigma, \lambda \rangle$ and a profile $(p(\omega), x(\omega))_{\omega \in A}$, an induced game Γ is a profile $\langle N, \Sigma, (\tilde{u}_i)_{i \in N} \rangle$. We extend the payoff function $\tilde{u}_i$ to the mixed strategy space where the expected utility is used. We denote by ρ_i a mixed strategy of player i.

Now, we present an equilibrium concept that reflects the idea of perfection.

Definition 2.4. *Given a mechanism $M = \langle \Sigma, \lambda \rangle$, $(\rho, (p(\omega), x(\omega))_{\omega \in A})$ is a* perfect market equilibrium (PME) *if*

(1) *for all* $\omega \in A$, $(p(\omega), x(\omega))$ *is a market equilibrium under* ω;
(2) ρ *is a Nash equilibrium of the induced game* $\langle N, \Sigma, (\tilde{u}_i)_{i \in N} \rangle$.

Given a PME, we sometimes call its on-path allocation a PME allocation. Analogously, we call its on-path object allocation a PME object allocation.

We also consider a refined concept of PME. The following concept of permutation independent PME requires that if the total endowments of the second stage are the same between the two outcomes of the first stage, then the equilibrium price vectors are the same. This reflects the idea of anonymity, i.e., changes in object holders would not change the price system as long as the total endowments are unchanged.[5]

Definition 2.5. $(\rho, (p(\omega), x(\omega))_{\omega \in A})$ *is a* permutation independent perfect market equilibrium (PIPME) *if it is a PME, and for all* $\omega, \hat{\omega} \in A$,

(PI): $|\omega| = |\hat{\omega}| \;\Rightarrow\; p(\omega) = p(\hat{\omega})$.

3. The Economy with Money

This section studies the economy with money. We assume that the set of available object types for player i, A_i, is equal to $\bar{O}$ for all $i \in N$. If money is available, then under the assumption of no income effect, utility becomes transferable, and the definition of Pareto optimality (resp. ω-optimality) is reduced to that of efficiency (resp. ω-efficiency).

3.1. **Existence.** First, we have the existence result of market equilibrium in the second stage due to Quinzii (1984).

Claim 3.1. *(Quinzii (1984)) For all* $\omega \in A$, *there exists at least one market equilibrium under* ω.

If there exists a market equilibrium under every ω, then by assigning a market equilibrium allocation under each ω, we can construct a game for the first stage. Then PME exists in the mixed strategy profile space since the existence of PME is reduced to the existence of Nash equilibrium. Thus, the following result is stated without proof.

Theorem 3.2. *There exists at least one PME.*

The quasi-linearity of the utility functions implies that the demand correspondence is independent of the initial allocation ω of the second stage, i.e., the object choice μ_i of player i is given by

$$\mu_i \in \arg\max_{a \in \bar{O}^\omega} \{v_i(a) - p_a\}. \tag{3.1}$$

For the subsequent analysis, the following lemma is useful.

Lemma 3.3. (1) *Given any* $\omega \in A$, *a market equilibrium allocation* (μ, m) *under* ω *is* ω-*efficient.*

[5] This definition also reflects the idea of price taking behavior in the sense that even if one changes his/her strategy in the assignment stage, it would not affect the second stage price system as long as the total endowments are unchanged. Indeed, in many mechanisms, including the Boston mechanism and the deferred acceptance algorithm, if there are a sufficient number of players who take undominated strategies, a unilateral deviation would not affect the total amount of objects available for the second stage market.

(2) *For any $\omega, \hat{\omega} \in A$ with $|\omega| = |\hat{\omega}|$, if (p, μ, m) is a market equilibrium under ω, then for any market equilibrium $(\hat{p}, \hat{\mu}, \hat{m})$ under $\hat{\omega}$, $\mu = \hat{\mu}$ holds, and there exists a market equilibrium under $\hat{\omega}$ of the form $(p, \mu, \hat{m})$, i.e., the price vector and the object allocation are the same.*

Proof.
Suppose (p, μ, m) is a market equilibrium under ω. First, we show ω-efficiency. Suppose the contrary, i.e., that there exists $\eta \in A^{\omega}$ such that $W(\eta) > W(\mu)$ holds.

For every player i, (3.1) implies

$$v_i(\mu_i) - p_{\mu_i} \geq v_i(\eta_i) - p_{\eta_i}. \tag{3.2}$$

From (3.2), we have

$$p_{\eta_i} - p_{\mu_i} \geq v_i(\eta_i) - v_i(\mu_i) \tag{3.3}$$

for all $i \in N$.

By taking the summation of the both sides across $i \in N$, (3.3) implies

$$\sum_{i \in N} \left[p_{\eta_i} - p_{\mu_i} \right] \geq W(\eta) - W(\mu) > 0. \tag{3.4}$$

Therefore, we have

$$\sum_{i \in N} p_{\eta_i} > \sum_{i \in N} p_{\mu_i}. \tag{3.5}$$

Rewriting the above inequality, we have

$$\sum_{a \in O} |\eta^a| p_a > \sum_{a \in O} |\mu^a| p_a. \tag{3.6}$$

This implies that there exists an object $a \in O$ such that $|\eta^a| > |\mu^a|$ and $p_a > 0$ hold. However, $|\mu^a| < |\eta^a| \leq |\omega^a|$ implies $p_a = 0$ by the equilibrium condition. This is a contradiction.

Next, suppose that (p, μ, m) and $(\hat{p}, \hat{\mu}, \hat{m})$ are market equilibria under ω and $\hat{\omega}$, respectively, such that $|\omega| = |\hat{\omega}|$ holds. Suppose the contrary, i.e., that $\mu \neq \hat{\mu}$ holds. By the genericity of v's, we have $W(\mu) \neq W(\hat{\mu})$. This is a contradiction to what we have proven above.

Finally, suppose that $|\omega| = |\hat{\omega}|$ holds, and that (p, μ, m) is a market equilibrium under ω. For each $i \in N$, let

$$\hat{m}_i = m_i - p_{\omega_i} + p_{\hat{\omega}_i}.$$

Given the price p, for each player i, the optimal object under $\hat{\omega}$ is μ_i since the demand correspondence does not depend on the initial endowment by Equation (3.1). Also, $\hat{m}_i$ is determined by player i's budget constraint. Thus, $(p, \mu, \hat{m})$ is a market equilibrium under $\hat{\omega}$. □

Using Lemma 3.3, we can construct a PIPME from a PME where the object allocations between the two equilibria are identical.

Corollary 3.4. *Suppose that there is a PME. Then, there exists at least one PIPME whose object allocation is identical to the PME object allocation.*

Proof. To prove this corollary, let us define the following. Given $\omega \in A$, let

$$\Omega_{\omega} = \{\hat{\omega} \in A \mid |\hat{\omega}| = |\omega|\}.$$

It is verified that these sets form equivalence classes. Let $\Omega = \{\Omega^1, \ldots, \Omega^L\}$ be a partition of A, i.e., $\Omega^{\ell} \cap \Omega^{\ell'} = \emptyset$ for $\ell \neq \ell'$ and $\cup_{i=1}^{L} \Omega^{\ell} = A$.

Suppose that $(\rho, (p(\omega), \mu(\omega), m(\omega))_{\omega \in A})$ is a PME. We construct a PIPME

$$(\rho^*, (p^*(\omega), \mu^*(\omega), m^*(\omega))_{\omega \in A})$$

as follows. Consider $\Omega = \{\Omega^1, \ldots, \Omega^L\}$. For each $\ell = 1, \ldots, L$, take an $\hat{\omega}^\ell \in \Omega^\ell$ in an arbitrary manner. Then for each $\ell = 1, \ldots, L$ and each $\omega \in \Omega^\ell$, let

$$\begin{aligned} p^*(\omega) &= p(\hat{\omega}^\ell), \\ \mu^*(\omega) &= \mu(\hat{\omega}^\ell), \\ m_i^* &= p^*_{\omega_i} - p^*_{\mu_i^*(\omega)},\ i \in N. \end{aligned}$$

Due to Lemma 3.3, $(p^*(\omega), \mu^*(\omega), m^*(\omega))$ is a market equilibrium under ω for all $\omega \in A$. This completes the construction of the second stage equilibrium profile $(p^*(\omega), x^*(\omega))_{\omega \in A}$ that satisfies (PI). Since the first stage strategy profile ρ^* is simply a Nash equilibrium of the induced game, the proof is completed. □

3.2. **Efficiency.** Next, we turn to the efficiency of PME. If there are not a sufficient number of players, we may not have efficiency. Let us consider the following example.

Example 3.1.

i	1	2
$v_i(\alpha)$	10	50
$v_i(\beta)$	20	5

TABLE 3.1. Inefficient PME

Suppose that the values together with N and O are given by Table 3.1, that $q^\alpha = q^\beta = 1$ *holds, and that* $1 >_a 2$ *holds for* $a = \alpha, \beta$. *Then, there is a PME where only* α *is consumed. We show it by construction. On the equilibriumm path of this PME, we let*

$$\omega = (\alpha, \phi),\ p = (p_\alpha, p_\beta) = (20, -),\ \mu = (\phi, \alpha),$$

and the utility gains of player 1 *and* 2 *are 20 and 30, respectively. Off the equilibrium path when* 2 *deviates to obtain the leftover* β *in the first stage, we let*

$$\omega = (\alpha, \beta),\ p = (p_\alpha, p_\beta) = (40, 10),\ \mu = (\beta, \alpha).$$

If this is the case, then the utility gain of player 1 is 50, while that of player 2 is 20. Therefore, 2 has no incentive to deviate. The lack of incentives to deviate in other off-paths is straightforward.

i	1	2	3
$v_i(\alpha)$	10	50	3
$v_i(\beta)$	20	5	3

TABLE 3.2. Inefficient PME disappears

If we add another player, 3, the situation changes even if this person is the lowest both in terms of priority and values. See Table 3.2. Also, suppose that $1 \succ_a 2 \succ_a 3$ *holds for* $a = \alpha, \beta$. *In this case, an inefficient equilibrium similar to the one that existed before in the two player case disappears. To see this, consider the corresponding allocation, i.e.,*

$$\omega = (\alpha, \phi, \phi),\ p = (p_\alpha, p_\beta) = (20, -),\ \mu = (\phi, \alpha, \phi).$$

This time, player 3 *obtains an object in neither stage. Thus, player* 3*'s surplus is zero even though* 3 *has a positive value for the objects. Thus,* 3 *has an incentive to take the leftover. The above allocation cannot be an equilibrium outcome.*

We generalize the argument of the above example to find a necessary and sufficient condition for efficiency. Given $\theta = 1, 2, \ldots$, let

$$V_\theta = \left\{ v \in \mathbb{R}^{N \times \bar{O}} \,\middle|\, \min_{a \in O} |\{i \in N | v_i(a) > 0,\ a \in A_i\}| = \theta \right\}$$

Objects are *scarce* if $v \in V_\theta$ with $\theta \geq 2Q - \min_{a \in O} q^a$ where $Q \equiv \sum_{a \in O} q^a$.

We are now in a position to state the following theorem.

Theorem 3.5. *The following two statements are equivalent:*

(1) *for all* $v \in V_\theta$*, a pure PME exists, and every pure PME allocation is efficient;*
(2) *objects are scarce, i.e.,* $\theta \geq 2Q - \min_{a \in O} q^a$.

Proof. [(1) ⇒ (2)]

We prove this direction by contraposition. Given q and $\succ$, assume $\theta < 2Q - \min_{a' \in O} q^{a'}$. We would like to show that there exist $v \in V_\theta$ and a pure inefficient PME. We let $|N| = \theta$. If $\theta < |N|$, we simply set the values of extra players to be negative.

Write $O = \{a_1, \ldots, a_{\bar{L}}\}$ in such a way that $q^{a_1} \geq q^{a_2} \geq \ldots \geq q^{a_{\bar{L}}}$ holds. Note $q^{a_{\bar{L}}} = \min_{a' \in O} q^{a'}$. In addition to $q^{a_1} < |N| < 2Q - q^{a_{\bar{L}}}$, $|N|$ must satisfy

$$\sum_{\ell=1}^{L-1} 2q^{a_\ell} \leq |N| < \sum_{\ell=1}^{L} 2q^{a_\ell} \tag{3.7}$$

for some $L = 1, \ldots, \bar{L}$ where the left hand side is zero if $L = 1$. Fix L. Let $O_L = \{a_1, \ldots, a_L\}$.

First, we identify the players who have higher priority for the objects in O_L than others. Let S_1 be the set of the top q^{a_1} players in terms of priority at a_1 among N, i.e., $S_1 = \{i_1, \ldots, i_{q^{a_1}}\}$ such that $i \succ_{a_1} j$ holds for all $i \in S_1$ and all $j \in N \setminus S_1$. Then sequentially define S_ℓ ($\ell = 2, \ldots, L-1$) as the set of the top q^{a_ℓ} players in terms of priority at a_ℓ among $N \setminus [\cup_{\ell'=1}^{\ell-1} S_{\ell'}]$. As for S_L, define it as the set of top $\min\{q^{a_L}, |N| - \sum_{\ell=1}^{L-1} 2q^{a_\ell}\}$ players in terms of priority at a_L among $N \setminus [\cup_{\ell'=1}^{L-1} S_{\ell'}]$. Let $S = \cup_{\ell=1}^{L} S_\ell$. Partition $N \setminus S$ into $B_1, \ldots, B_L$ in such a way that $|B_\ell| = q^{a_\ell}$ holds for $\ell = 1, \ldots, L-1$, and $B_L = N \setminus [S \cup_{\ell=1}^{L-1} B_\ell]$. Note that S_L may be empty, that $|S_L| < q^{a_L}$ implies $B_L = \emptyset$, and that $|B_L| < q^{a_L}$.

Now, we construct v as follows:
for $i \in S_\ell$ ($\ell = 1, \ldots, L-1$), let $v_i(a)$ be any number in $(0, 1)$ for all $a \in O$;
for $i \in B_\ell$ ($\ell = 1, \ldots, L-1$), let $v_i(a)$ be any number satisfying the following:

$$v_i(a) \in \begin{cases} (14, 15) & \text{if } a = a_\ell, \\ (0, 1) & \text{otherwise.} \end{cases}$$

In doing so, the numbers are chosen in such a way that (2.1) holds, i.e., genericity is guaranteed. As for the players in S_L and B_L, we define v separately as follows. Partition S_L into S'_L and S''_L with $|S'_L| = |B_L|$, and
for $i \in S'_L$, let $v_i(a)$ be any number in $(0, 1)$ for all $a \in O$,
for $i \in S''_L \cup B_L$,

$$v_i(a) \in \begin{cases} (2,3) & \text{if } a = a_L, \\ (0,1) & \text{otherwise.} \end{cases}$$

Note that S'_L is empty if B_L is empty.

Next, in the first stage, let

$$\omega_i^* = \begin{cases} a_\ell & \text{if } i \in S_\ell, (\ell = 1, \dots, L), \\ \phi & \text{if } i \in B_\ell, (\ell = 1, \dots, L). \end{cases}$$

To attain this profile under DA, they put what they are supposed to obtain at the top of the list to submit. To attain it under Boston, they choose their respective objects in the first round.

In the second stage, let $p^*_{a_\ell} = 10 - \ell/L$ for all $\ell = 1, \dots, L-1$ and $p^*_{a_L} = 1$ under ω^*. This price vector does not change if $i \in S$ unilaterally deviates. Let p_{a_ℓ} $(\ell = 1, \dots, L)$ change from $p^*_{a_\ell}$ to $p^*_{a_\ell} + 1$ if $i \in B_\ell$ unilaterally deviates. Assign equilibrium price and allocation vectors to other endowment profiles, i.e., nodes that are reached only when two or more players deviate. A typical behavior pattern and prices on the path is given in Table 3.3.

	a_1	$\cdots$	a_{L-1}	a_L	leftovers
Buyers	$\overbrace{\spadesuit\,\spadesuit\cdots\spadesuit}^{B_1}$	$\cdots$	$\overbrace{\heartsuit\,\heartsuit\cdots\heartsuit}^{B_{L-1}}$	$\overbrace{\clubsuit\,\clubsuit}^{B_L}$	
Sellers	$\underbrace{\spadesuit\,\spadesuit\cdots\spadesuit}_{S_1}$	$\cdots$	$\underbrace{\heartsuit\,\heartsuit\cdots\heartsuit}_{S_{L-1}}$	$\underbrace{\clubsuit\,\clubsuit}_{S'_L}\ \underbrace{\clubsuit\cdots\clubsuit}_{S''_L}$	$\square\,\square\,\cdots$
price p^*	$10 - \frac{1}{L}$	$\cdots$	$10 - \frac{L-1}{L}$	1	

TABLE 3.3. A typical behavior pattern and prices on the path

We then show that the profile described above constitutes a PME. Observe first what the players do on the path:

- player $i \in S_\ell$ $(\ell = 1, \dots, L-1)$ obtains object a_ℓ in the first stage, sells it to a player in B_ℓ at $p^*_{a_\ell}$ in the second stage, and gains $p^*_{a_\ell} = 10 - \ell/L$ through the two stage activity;
- player $i \in B_\ell$ $(\ell = 1, \dots, L-1)$ obtains ϕ, the null object, in the first stage, buys a_ℓ in the second stage, and gains $v_i(a_\ell) - p^*_{a_\ell} \in (4, 6)$;
- player $i \in S'_L$ obtains object a_L in the first stage, sells it to a player in B_L at $p^*_{a_L} = 1$, and gains 1;
- player $i \in S''_L$ obtains object a_L in the first stage, consumes it, and gains $v_i(a_L) \in (2, 3)$;
- player $i \in B_L$ obtains ϕ in the first stage, buys a_L at $p^*_{a_L} = 1$, and gains $v_i(a_L) - 1 \in (1, 2)$.

Next, we check what they obtain if they make a unilateral deviation:

- player $i \in S_\ell$ $(\ell = 1, \dots, L-1)$ may obtain $a_{\ell'}$ for some $\ell' > \ell$ to gain $p^*_{a_{\ell'}}$, $v_i(a) \in (0, 1)$ for some $a \in \bar{O}$; all of them are less than $p^*_{a_\ell}$;

- player $i \in B_\ell$ ($\ell = 1, \ldots, L-1$) may obtain a leftover, say, a in the first stage and sell it and buy a_ℓ at the same time in the second stage. By doing so, he/she gains at most $v_i(a_\ell) + p_a - (p^*_{a_\ell} + 1)$ due to a price increase of object a_ℓ. Since p_a is at most one, the gain is less than or equal to $v_i(a_\ell) - p^*_{a_\ell}$;
- since the prices of object a_L and other objects are at most one, player $i \in S'_L$ gains at most one;
- player $i \in S''_L$ gains either one or $v_i(a_L) - 1$;
- player $i \in B_L$ may obtain a leftover, say, a in the first stage and sell it and buy a_L at the same time in the second stage. By doing so, he/she gains at most $v_i(a_L) + p_a - (p^*_{a_L} + 1)$ due to a price increase of object a_L. Since p_a is at most one, the gain is less than or equal to $v_i(a_L) - p^*_{a_L}$.

Note that the price of the object other than $a_1, \ldots, a_L$ is at most one due to the construction of v. Hence, none of them has an incentive to make a unilateral deviation. Since there is a leftover that would have induced a positive utility gain, the PME is not efficient.

[(1) $\Leftarrow$ (2)]
Assume $\theta \geq 2Q - \min_{a \in O} q^a$.
Existence:
Let ω be an allocation with no leftover, i.e., $|\omega| = q$. Suppose (p^*, μ^*, m^*) is an ME under ω (such an ME exists). We may assume $p^*_a > 0$ for $a \in O$ since there is a sufficient amount of demand for each $a \in O$. For any ω' with no leftover, i.e., $|\omega'| = q$, let $p(\omega') = p^*$. Adjusting m' appropriately, we obtain an ME (p^*, μ^*, m') under ω' due to Lemma 3.3.

Align the objects $O = \{a_1, a_2, \ldots, a_L\}$ in such a way that $p_{a_1} \geq p_{a_2} \geq \ldots, \geq p_{a_L}$ holds. Then, as in the proof of the other direction, let S_1 be the set of the top q^{a_1} players in terms of priority at a_1 among N, i.e., $S_1 = \{i_1, \ldots, i_{q^{a_1}}\}$ such that $i \succ_{a_1} j$ holds for all $i \in S_1$ and all $j \in N \setminus S_1$. Then sequentially define S_ℓ ($\ell = 2, \ldots, L$) as the set of the top q^{a_ℓ} players in terms of priority at a_ℓ among $N \setminus [\cup_{\ell'=1}^{\ell-1} S_{\ell'}]$. Let $S = \cup_{\ell=1}^{L} S_\ell$.

Next, in the first stage, let

$$\omega^*_i = \begin{cases} a_\ell & \text{if } i \in S_\ell, (\ell = 1, \ldots, L), \\ \phi & \text{if } i \in N \setminus S. \end{cases}$$

To attain this profile under DA, the players in S put what they are supposed to obtain at the top of the list to submit. The players in $N \setminus S$ submit the truth telling strategies. To attain it under Boston, the players in S choose their respective objects in the first round, and the players in $N \setminus S$ choose some objects in O instead of ϕ.

Along the path, each player $i \in N$ obtains ω^*_i in the first stage. Moreover, even if one, say, player i, makes a unilateral deviation, there would be no leftover since there are players in $N \setminus S$ waiting for any leftover.

This strategy profile constitutes a pure PME along with ME's mentioned above (and appropriately chosen ME's for other ω's).

Efficiency:
Take either DA or Boston as the first stage mechanism. The following proof is the same for both. Note that we have assumed $\theta \geq 2Q - \min_{a \in O} q^a$. Take any $v \in V_\theta$. Suppose $a \in O$ has some

leftover, i.e., $|\omega^a| < q^a$. Observe that at least q^a players who cannot obtain $b \in O \setminus \{a\}$ in neither stage and have a positive value for a. Let W be the set of such agents. Note $|W| \geq q^a > |\omega^a|$. Then $p_a \geq \min_{i \in W} v_i(a) > 0$ holds; for if not, there would be excess demand for a. Then there exists $\ell \in W$ who obtains nothing in the first stage, i.e., $\omega_\ell = \phi$. This agent ℓ has an incentive to obtain the leftover to gain $v_\ell(a)$ instead of $\max\{v_\ell(a) - p_a, 0\}$. Thus, there is no leftover. Once this is established, we resort to the efficiency property of the second stage market equilibrium to assure efficiency.

□

4. College Admission and Labor Market

We consider a decentralized labor market after college admission. There are two sets of players. N_s is the set of students. N_f is the set of firms. We have $N = N_s \cup N_f$ and $N_s \cap N_f = \emptyset$. College degrees are objects. A firm can demand a degree only if it is owned by some student.[6] Every student selects a college (including not going to college, corresponding to ϕ), taking into account the future job prospect. For all $i \in N_s$, the set of available object types for player i in the first stage is $A_i = \bar{O}$. On the other hand, for all $i \in N_f$, the set of available object types for player i in the first stage is $A_i = \{\phi\}$.

We assume the following.

Condition 4.1.

(Zero) for N_s: $v_i(a) = 0$ *holds for all* $i \in N_s$ *and all* $a \in \bar{O}$,

In the presence of (Zero) for N_s, we assume genericity only for N_f. Assumption (Zero) for N_s implies that the firms, not the students, intrinsically demand the college degrees.

Given $N_0 \subset N$, let $A_{N_0} = \{\omega \in A | i \in N \setminus N_0 \to \omega_i = \phi\}$. The following existence result is similar to that in the previous section.

Lemma 4.1. *Assume (Zero) for N_s. Given $\omega \in A$, there exists at least one market equilibrium under ω.*

Then PME exists in the mixed strategy since the existence of PME is reduced to the existence of Nash equilibrium. The following result is stated below without proof.

Theorem 4.2. *Assume (Zero) for N_s. Then there exists at least one PME.*

We have the following corollary that corresponds to Corollary 3.4.

Corollary 4.3. *Assume (Zero) for N_s. Suppose that there is a PME. Then there exists at least one PIPME whose object allocation is identical to the PME object allocation.*

The proof of this corollary is essentially the same as that of Corollary 3.4.

4.1. **Efficiency.**
In order to state the subsequent result, we need to modify the definition of scarcity from what we have in the previous section. Given $\theta = 1, 2, \ldots$, let

$$V_\theta^f = \left\{ v \in \mathbb{R}^{N \times \bar{O}} \,\middle|\, \min_{a \in O} |\{i \in N_f | v_i(a) > 0\}| = \theta \right\}$$

[6] We do not consider signaling effects here.

Objects are *scarce'* if

$$|N_s| \geq Q \text{ and } v \in V_\theta^f \text{ with } \theta > Q.$$

The next theorem corresponds to Theorem 3.5.

Theorem 4.4. *Assume (Zero) for N_s. Then the following two statements are equivalent:*

(1) *for all $v \in V_\theta$, a pure PME exists, and every pure PME allocation is efficient;*
(2) *objects are scarce', i.e., $|N_s| \geq Q$ and $v \in V_\theta^f$ with $\theta > Q$.*

In the following proof of the above theorem, we skip some details when they are similar to what we have stated in the proof of Theorem 3.5.

Proof. ($\Rightarrow$) Suppose that objects are *not* scarce', i.e., either $|N_s| < Q$ or $v \in V_\theta^f$ with $\theta \leq Q$ (or both). If $|N_s| < \theta$ holds, then efficiency is trivially violated as there are not sufficiently many students who deliver all the objects to the firms that need them.

Therefore, assume $\theta \leq |N_s|$. We construct v as follows. Align the objects in an arbitrary manner, $O = \{a_1, \ldots, a_{\bar{L}}\}$. There exists a unique $L = 1, \ldots, \bar{L}$ such that $q_{a_1} + \cdots + q_{a_{L-1}} < \theta \leq q_{a_1} + \cdots + q_{a_L}$ holds. Fix L. Let $\hat{N}_f \subset N_f$ satisfy $|\hat{N}_f| = \theta$ and $\forall i \notin \hat{N}_f \forall a \in O[v_i(a) < 0]$. Then assign a number to each $v_i(a)$ ($i \in \hat{N}_f, a \in O$) in such a way that for each $\ell = 1, \ldots, \bar{L} - 1$, and for all $i, j \in \hat{N}_f$, $v_i(a_\ell) > v_j(a_{\ell+1}) > 0$ holds.

Let μ^* be the efficient object allocation given v. It must be the case that $|\mu^{*a}| = q^a$ for $a = a_1, \ldots, a_{L-1}$ and that $0 < |\mu^{*a_L}| \leq q^{a_L}$. Consider ω with $|\omega| = |\mu^*|$. Then (p, μ^*, m) becomes an ME under ω for some p and m. It is verified, due to the way we construct v, that $p_{a_1} \geq p_{a_2} \geq \ldots \geq p_{a_L}$. Then there is another ME, denoted (p^*, μ^*, m^*), such that $p^*_{a_\ell} = p_{a_\ell} - p_{a_L}$ holds for all $\ell = 1, \ldots, L$. Note that $p^*_{a_L} = 0$ holds.

Assign the objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using $\succ$, i.e., those who have higher priority at a_1 obtain a_1, and so on. As for a_L, assign the objects to the remaining students so that the total number of the students assigned to some tangible objects becomes θ. Assign the other students to ϕ. Denote this assignment profile ω^*. Let (p^*, μ^*, m') be the ME under any ω' with $|\omega'| = |\omega^*|$. The existence of such an ME is proven in the same manner as in the proof of Corollary 4.3.

Remove one player, say, i from ω^{*a_L} to obtain ω^{**}. We would like to have this ω^{**} as the PME allocation of the first stage. Let us check if there is no incentive to deviate. Under ω^{**}, there is one firm that cannot buy a tangible object in the second stage, and there is at least one student who does not obtain a leftover in the first stage. If such a student obtains the object, then the first stage object allocation becomes ω^* (or some ω' with $|\omega'| = |\omega^*|$ to be precise), and therefore, the price of the object this student obtains is zero. Thus, the student has no incentive to deviate in the first stage. An inefficient outcome arises as a PME allocation.

($\Leftarrow$) Suppose that objects are scarce, i.e., $|N_s| \geq Q$ and $v \in V_\theta^f$ with $\theta > Q$. Take v as given along with other parameters, $\succ$ and q.

We show existence first. Take some ω with $|\omega| = q$. Let (p^*, μ^*, m) be an ME under ω. Align $O = \{a_1, \ldots, a_L\}$ in such a way that $p^*_{a_1} \geq p^*_{a_2} \geq \ldots \geq p^*_{a_L}$ holds. Since $\theta > Q$ holds, there exists $j \in N_f$ such that $\mu^*_j = \phi$ and $v_j(a_L) > 0$ hold. Therefore, $p^*_{a_L} \geq v_j(a_L) > 0$. Assign objects to the players in N_s in the first stage from a_1 to a_{L-1} to fill their respective quotas, using $\succ$. We can do it as $|N_s| \geq Q$. Assign the other students to ϕ. Denote this assignment profile ω^*. Under ω^*,

(p^*, μ^*, m^*) becomes an ME for some m^*. Let ω^* be the outcome of the first stage. Then together with appropriate off-path ME's, we have a PME as nobody has an incentive to deviate.

We next prove efficiency. Suppose that $(\sigma, (p(\omega), \mu(\omega), m(\omega)))$ is a PME. Let $\omega^* = \lambda(\sigma)$. Take any ω. Since $\theta > Q$ holds, for all $a \in O$, there exists $j \in N_f$ such that $\mu_j(\omega) = \phi$ and $v_j(a) > 0$ hold. Therefore, $p_a(\omega) \geq v_j(a) > 0$ for all $a \in O$; otherwise, j would buy a in ME. Suppose that $a \in O$ has some leftover, i.e., $|\omega^{*a}| < q^a$. Since $|N_s| \geq Q$, there exists at least one student who does not obtain any tangible object. This player has an incentive to obtain the leftover a since under any ω, $p_a(\omega) > 0$ as we have shown.

□

5. The economy without Money

This section considers the economy without money. Also, we assume that all goods are valuable for all the players, i.e., for all i in N, for all $a \in O$ $v_i(a) > 0$ holds. In this section, we assume that the quota of each object in O is one.

For convenience, we summarize some of the assumptions in the following.

Condition 5.1.

(+Value): *for all i in N and for all $a \in O$, $v_i(a) > 0$,*
(Quota1): *for all $a \in O$, $|q^a| = 1$.*

Under (+Value), all tangible objects have positive intrinsic values for all. The set of positive value profile is:

$$V_+ = \{v \in \mathbb{R}^{\bar{O}\times N} | \forall i \in N \forall a \in O \ v_i(a) > 0\}.$$

5.1. Existence and optimality.

The condition for the existence of market equilibrium in the second stage is non-trivial in the case of no money. Shapley and Scarf (1974) essentially shows that for any initial endowment, a market equilibrium exists if all the tangible objects have a positive value for everyone, and if the quota of each object is one.

Lemma 5.1. *Assume (+Value), and (Quota1). For all $\omega \in A$, market equilibrium exists under ω.*

Proof.
Assume (+Value), and (Quota1). Shapley and Scarf (1974) shows that there is a sequence of top trading cycles $S_1, \ldots, S_L$ where $S_1 \neq \emptyset$ is a top trading cycle in N, $S_{\ell+1} \neq \emptyset$ is a top trading cycle in $N \setminus \cup_{\ell'=1}^{\ell} S_{\ell'}$ ($\ell = 1, \ldots, L-1$), and $\cup_{\ell=1}^{L} S_\ell = N$ (see Appendix A.2 for the definition of top trading cycles). Next, Shapley and Scarf (1974) attaches, in the present notation, a price p_ℓ to each good held by a player in S^ℓ ($\ell = 1, \ldots, L-1$) in such a way that we have

$$p^1 > \cdots > p^L > 0.$$

Then, the price system defined above constitutes a competitive price system.

We let p_1, the highest price, not exceed $\min_{i\in N} \min_{a,b\in\bar{O},\ a\neq b}[v_i(a) - v_i(b)]$, which is positive due to genericity. Then, no player has an incentive to deviate in the second stage under ω. □

Using this claim and the existence result of subgame perfect equilibrium for a finite game, we have the existence result for PME, which is stated without proof[7].

[7]Note that PIPME may not exist in the case of no money.

Theorem 5.2. *Assume (+Value), and (Quota1). Then, there exists at least one PME.*

Next, we show that a market equilibrium allocation is "optimal" given the initial endowment.

Lemma 5.3. *Assume (+Value), and (Quota1). Suppose that given ω, μ is a market equilibrium object allocation. Then, μ is ω-optimal.*

Since the proof of this lemma is an application of the standard proof of the first fundamental theorem of welfare economics, we relegate it to the appendix.

Proof. See Appendix B.1. □

Theorem 5.4. *Assume (+Value), and (Quota1). The following two statements are equivalent.*

(1) *$|\{i \in N | A_i = \bar{O}\}| \geq Q$ or for all $i \in N$ $A_i = \bar{O}$.*
(2) *Any pure PME object allocation is Pareto optimal.*

Proof.

[(1) ⇒ (2)]
Suppose $|\{i \in N | A_i = \bar{O}\}| \geq Q$. Suppose ω is the PME outcome of the first stage. Then, $|\omega| = q$ holds. Suppose the contrary, i.e., there exists a leftover a. Then, there exists i s.t. $\omega_i = \phi$, $A_i = \bar{O}$ and $v_i(a) > 0$. Then, this i cannot obtain any object in O in the second stage because the prices of all the objects in O are strictly positive. For if not, the object i can obtain has a zero price, which implies excess demand. Thus, this player i has an incentive to obtain a in the first stage. This is a contradiction.

Next, suppose for all $i \in N$ $A_i = \bar{O}$.

Let $\mu \in A^+$ be a PME allocation, and let p be the price profile in this PME. Suppose the contrary, i.e., that there exists $\eta \in A^+$ that Pareto dominates μ. Partition N into N_e and N_d where we have

$$v_i(\eta_i) = v_i(\mu_i) \quad \text{if} \quad i \in N_e,$$
$$v_i(\eta_i) > v_i(\mu_i) \quad \text{if} \quad i \in N_d.$$

Note $N_d \neq \emptyset$. Since there is no indifferent object other than itself, $\eta_i = \mu_i$ holds for all $i \in N_e$. Take any $i_0 \in N_d$. Player i_0 would have obtained η_{i_0} if it were available in either stage. In the first stage, therefore, it must be the case that another player in N_d who obtained η_{i_0} under μ; otherwise, player i_0 could have obtained it directly in the first stage. Also, player i_0 could have obtained it in the second stage if $p_{\eta_{i_0}} \leq p_{\mu_{i_0}}$. But, repeating the same proof as the one in Lemma 5.3, we prove this would lead to a contradiction.

[(2) ⇒ (1)] This is a proof by the contraposition. Suppose that $|\{i \in N | A_i = \bar{O}\}| < Q$ holds, and that there exists $i \in N$ with $A_i = \{\phi\}$. Then, there exists a leftover that could have been consumed by some player with a positive value. Thus, the PME allocation is not Pareto optimal.

□

Proposition 5.5. *Assume (+Value) and (Quota1). PME exists, and in any PME allocation, there is no player with $A_i = \{\phi\}$ who obtains an object in O.*

Proof. Suppose the contrary, i.e., in some PME allocation, there is a player with $A_i = \{\phi\}$ who can obtain an object in O in the second stage. Then, as we see in the proof of the theorem 5.4, that object has a zero price. Due to the assumption (+Value), this leads to an excess demand. This is a contradiction. □

5.2. **Quotas and values.**
Existence is not guaranteed if the quota exceeds one for some object type as the next example shows.

Example 5.1.

	1	2	3
α	10	20	20
β	20	10	10

TABLE 5.1. Value

Let the values of this economy be given in Table 5.1. Suppose

$$\omega = (\alpha, \beta, \beta).$$

Then we have no market equilibrium in the second stage under ω. To begin with, we have $p_\alpha \leq 20$ and $p_\beta \leq 10$. For if not, there would be excess supply with a positive price. Consider two cases. First, suppose $p_\alpha \leq p_\beta \leq 10$. Then both 2 and 3 can afford α, and therefore, the demand for α is at least two, which leads to excess demand as there is only one unit of object α. Second, suppose $p_\alpha > p_\beta$. Then no player demands α, which leads to excess supply for α with a positive price. Thus, no market equilibrium exists.

Also, existence is not guaranteed if (+Value) in Assumption 5.1 is violated.

Example 5.2.

	1	2	3
α	20	−10	20
β	10	−20	10

TABLE 5.2. Value

Let the values of this economy be given in Table 5.2. Suppose

$$\omega = (\phi, \alpha, \beta).$$

Then we have no market equilibrium in the second stage under ω. Suppose the contrary, i.e., that p is a market equilibrium price. First, we have $p_\phi = 0$. Next, we would like to show $p_\alpha = 0$. Suppose not, i.e., $p_\alpha > 0$. Then there must be a positive demand for α, which occurs only if $p_\beta \geq p_\alpha > 0$ since 3 must demand α. This implies that there is no demand for β since 1 has neither money nor object with a positive price. This is a contradiction. Thus, $p_\alpha = 0$ holds. But, this would induce the excess demand for α. Hence, no market equilibrium exists.

5.3. **Stability and Market Equilibrium.** We define the concept of stable market equilibrium (SME), which is a market equilibrium of which object allocation is stable. This subsection studies the relationship between SME and PME. SME requires that the object allocation of a market equilibrium should be stable. It considers neither the incentive in the first stage nor off-the-path market equilibria of the second stage. Therefore, while it is easy to verify some allocation is an SME allocation, it is not clear if the players really follow this equilibrium. On the other hand, PME takes into account all the incentives, both on and off-the equilibrium path, and in general, it is hard to characterize.

The stability of object allocations is also defined in the standard manner.

Definition 5.1. *An object allocation $\mu \in A$ is* stable *if*

- $\forall i \in N, \forall j \in N\ [\mu_j \in O \wedge i \succ_{\mu_j} j \Rightarrow v_i(\mu_i) \geq v_i(\mu_j)]$,
- $\forall a \in \bar{O}\ \forall i \in N\ [|\mu^a| < q^a \Rightarrow v_i(\mu_i) \geq v_i(a)]$.

Definition 5.2. *Given $v \in V_+$ and $\succ$, $(p, \mu) \in \mathbb{R}_+^{\bar{O}} \times A$ is a* stable market equilibrium *(SME) if*

- *(p, μ) is a market equilibrium under μ itself;*
- *μ is stable.*

In order to further study the solution concepts, we introduce the concept of priority cycle as stated in Ergin (2002)[8].

Definition 5.3. *Let $\succ$ be a priority structure and q be a quota profile.* A priority cycle *is constituted of distinct $a, b \in O$ and $i, j, k \in N$ such that the following is satisfied:*
(C) Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

Definition 5.4. *Let $\succ$ be a priority structure and q be a vector of quotas. A* generalized cycle of priority *is constituted of distinct $a_1, a_2, \ldots, a_n \in O$ and $i, k_1, \ldots, k_n \in N$ such that the following are satisfied:*
(C') Cycle condition: $k_1 \succ_{a_1} i \succ_{a_1} k_n \succ_{a_n} k_{n-1} \succ_{a_{n-1}} k_{n-2} \ldots k_2 \succ_{a_2} k_1$.

If $\succ$ has a generalized cycle, then it also has a cycle. However, this assertion can be shown in the same way as in Ergin (2002). If the priority structure is not cyclical, it is called *acyclical*. The following proposition states the existence of SME.

Lemma 5.6. *Assume (+Value), and (Quota1). SME exists if the priority structure is acyclical.*

Proof. See Appendix B.2. □

Theorem 5.7. *Assume $|O| \geq 3$, (Quota1).*
Then the following two are equivalent:

- *For any $A = (A_i)_{i \in N}$ and for all $v \in V_+$, an SME (p, μ) exists, and μ is a PME allocation*
- *$\succ$ is acyclical*

Proof. See Appendix B.3. □

[8]The definition of the cycle and acyclicity are different from that of Ergin (2002) in that Ergin (2002) includes the condition on scarcity in the definition as well

Appendix A. Mechanisms and Trading Cycles

A.1. **Properties of the truth-telling strategies of (DA).**

Definition A.1. *Assume (DA). Then, ζ^* is a truth-telling strategy if for all $i \in N$ with $A_i = \bar{O}$ and $a, a' \in \bar{O}$, $v_i(a) > v_i(a')$ implies that a is ranked higher than a' in the list of objects.*

Lemma A.1. *Assume (+Value), (Quota1) and (DA). Let $\eta \in A$ be the allocation by the truth-telling strategy. Suppose also that $\mu \in A$ with $\mu \neq \eta$ is stable. Then, μ is Pareto dominated by η.*

Proof.
Assume (+Value) and (Quota1). Suppose not, i.e., that a stable object allocation $\mu \neq \eta$ is not Pareto dominated by η. Then under genericity, there exists $k_1 \in N$ such that $v_{k_1}(\mu_{k_1}) > v_{k_1}(\eta_{k_1})$ holds. Take such a player k_1. Let $a = \mu_{k_1}$. In DA algorithm with ζ^*, in some step t_1, k_1 is rejected at a. Therefore, there exists $k_2 \in N$ s.t. $k_2 >_a k_1$ and k_2 comes to a at t_1. Note that k_2 is not in a under μ, i.e., $\mu_{k_2} \neq a$. Then, the stability of μ implies that $v_{k_2}(\mu_{k_2}) > v_{k_2}(a)$ holds.

Again, ζ^* implies that k_2 must be rejected at μ_{k_2} before t_1.

In this way, we can construct a sequence of players $\{k_n\}_{n=1}^{\infty}$ such that for each $n = 3, 4, \ldots$, there exists a step t_n such that $t_n < t_{n-1}$ and k_n is rejected at μ_{k_n}. This is a contradiction since there are finitely many steps in DA algorithm. □

The following proposition is a direct consequence of the above lemmata, which is stated without a proof.

Proposition A.2. *Assume (+Value) and (Quota1). Suppose for all $i \in N$, $A_i = \bar{O}$. Suppose $\mu \in A$ is stable and Pareto optimal. Also, suppose $\eta \in A$ is the allocation by the truth-telling strategy of (DA). Then, $\mu = \eta$.*

A.2. **Top trading cycles.**

We define the top trading cycles due to Shapley and Scarf (1974) in this appendix.[9]

Definition A.2. *Assume (+Value), and (Quota1). Consider an object allocation $\mu \in A^+$.*

The following is the top trading cycles *with an initial object μ.*

Given $N' \subset N$ and $\mu \in A^+$, we define a trading cycle *among N' under μ as a nonempty subset S of N', whose $K-1$ members can be indexed in a cyclic order: $S = \{i_1, i_2, \ldots, i_{K-1}\}$ with $i_K = i_1$, in such a way that each trader i_k ($k = 1, \ldots, K-1$) weakly prefers $\mu_{i_{k+1}}$ to μ_j for all $j \in N'$.*

We then define the following algorithm.

Step0: Assign $\mu \in A^+$ to the players in N. Let $N_1 = N$. Let p_0 be any positive number.

Step $t (t \geq 1)$: There is at least one trading cycle among N_t under μ. Take one of them and denote it S_t, which may be a singleton. Let $N_{t+1} = N_t \setminus S_t$. Let the price of all the objects held by the players in S_t be p_t satisfying $p_t < p_{t-1}$.

Stop when N_{t+1}. Otherwise, go to Step $t + 1$.

Note that the above algorithm is terminated in a finite number of steps since at least one player is removed from the mechanism in each step. Since for all i in N, for all a in O, $v_i(a) > 0$ holds, ϕ is never chosen by any player until all the objects in O are removed.

---

[9] See also Kesten (2006) and Piccione and Rubinstein (2007).

Appendix B. Proofs

B.1. **Proof of Lemma 5.3.**

Proof.
Since μ is a market equilibrium object allocation, there exists $p \in \mathbb{R}_+^O$ with (p, μ) being a market equilibrium under ω. Suppose the contrary that there exists $\eta \in A^\omega$ such that η Pareto dominates μ. Partition N into N_e and N_d where we have

$$v_i(\eta_i) = v_i(\mu_i) \quad \text{if} \quad i \in N_e,$$
$$v_i(\eta_i) > v_i(\mu_i) \quad \text{if} \quad i \in N_d.$$

Note $N_d \neq \emptyset$. Since there is no indifferent object other than itself, $\eta_i = \mu_i$ holds for all $i \in N_e$. Take any $i_0 \in N_d$. Player i_0 could have obtained η_{i_0} in the second stage if $p_{\eta_{i_0}} \le p_{\mu_{i_0}}$. Therefore, we must have

$$p_{\eta_{i_0}} > p_{\mu_{i_0}}.$$

There exists $i_1 \in N_d$ who obtained η_{i_0} under μ, i.e., $\mu_{i_1} = \eta_{i_0}$. Thus, we have

$$p_{\mu_{i_1}} > p_{\mu_{i_0}}.$$

We repeat the same procedure to construct a sequence $(i_0, i_1, i_2, \ldots)$ with

$$p_{\mu_{i_{k+1}}} > p_{\mu_{i_k}}, \quad k = 0, 1, 2, \ldots, \tag{B.1}$$

until the same player reappear along the sequence, i.e., $i_K = i_L$ for some $K < L$. Adding (B.1) from $k = K$ to $k = L - 1$, we obtain

$$\sum_{k=K}^{L-1} p_{\mu_{i_{k+1}}} > \sum_{k=K}^{L-1} p_{\mu_{i_k}}. \tag{B.2}$$

The both sides are the same since $i_L = i_K$ holds. This is a contradiction.

□

B.2. **Proof of Lemma 5.6.**
Before proving Lemma 5.6, we state and prove the following lemma.

Lemma B.1. *Assume (+Value), and (Quota1) . If $\mu \in A^+$ is Pareto optimal, all the players in the trading cycle mechanism with an initial object μ are in a trading cycle as a singleton.*

Proof.
Assume (+Value), and (Quota1). Suppose not, i.e., there is a trading cycle $(i_1, \ldots, i_K)$ with $K > 1$. Then, $v_{i_k}(\mu_{i_{k+1}}) > v_{i_k}(\mu_{i_k})$ and $v_{i_K}(\mu_{i_1}) > v_{i_K}(\mu_{i_K})$ hold. This implies that we can construct $\eta \in A^+$ such that for all $j \in N \setminus \{i_1, \ldots, i_K\}$, $\eta_j = \mu_j$, for all $k = 1, \ldots, K-1$, $\eta_{i_k} = \mu_{i_{k+1}}$ and $\eta_{i_K} = \mu_{i_1}$. Then, η Pareto dominates μ. This is a contradiction to the assumption. □

Now, the proof of Lemma 5.6 is provided. Assume (+Value), and (Quota1). Remove $j' \in N$ with $A_{j'} = \{\phi\}$. Thus, suppose N means those players j with $A_j = \bar{O}$. Also, suppose that the priority structure is acyclical. Then, Ergin (2002) shows that the truth-telling DA object allocation μ becomes a stable and Pareto optimal allocation.

First, we assign a price to each object a using the trading cycle mechanism with an initial object μ (see Appendix A.2 for the definition of the trading cycle mechanism). Let $(p_a)_{a\in O}$ be such a constructed price where p_1, the highest price, not exceed

$$\min_{i\in N} \min_{a,b\in\bar{O},\ a\neq b} [v_i(a) - v_i(b)],$$

which is positive due to genericity. Then, for all $i \in N$, for all $a \in O$ if $p_{\mu_i} \geq p_a$ holds with $a \neq \mu_i$, then $v_i(\mu_i) > v_i(a)$ holds. Also, no player has an incentive to deviate in the second stage under μ. Since Lemma B.1 implies that every player forms a trading cycle as a singleton, the object allocation after the trading cycle mechanism with an initial object μ is μ. Therefore, (p, μ) is a market equilibrium under μ itself.

□

B.3. **Proof of theorem 5.7.**

B.3.1. *(DA).*

Lemma B.2. *Assume (DA). Assume (p, μ) is a SME. Then, μ is a PME object allocation of truth-telling strategy $\Longleftrightarrow$ μ is a PME object allocation.*

Proof. Necessity is trivial.
Proof of sufficiency.

Suppose μ is a PME object allocation. Since a SME (p, μ) exists, then, Proposition A.2 implies that μ is a PME object allocation of truth-telling strategy. □

By the lemma B.2, the following proof of sufficiency considers a PME object allocation of truth-telling strategy.

Proof.
Proof of sufficiency:

Assume (Quota1), and (DA). Assume there is no cycle in the priority. Suppose the contrary, i.e., there exists A and $v \in V_+$ s.t. SME object allocation μ is not PME object allocation.

We want to show there exists a priority cycle.

First, remove j' with $A_{j'} = \{\phi\}$ from the economy. Hereafter, N means those players j with $A_j = \bar{O}$. $\succ$ and v are reduced to N as well.

Let $(p(\omega), \mu(\omega))_{\omega\in A}$ be a profile of market equilibrium that satisfies $(p(\mu), \mu(\mu)) = (p, \mu)$. Also, let $\zeta^* = (\zeta^*_j)_{j\in N}$ be the profile of truth-telling strategy.

We consider $\hat{\rho}$ that puts probability one on this ζ^*. Then, the outcome of the first stage is μ with probability one under $\hat{\rho}$.

We want to show that $(\hat{\rho}, (\hat{p}(\omega), \hat{\mu}(\omega))_{\omega\in A})$ is PME. Suppose not, i.e., that there exists $i \in N$ with $\rho'_i \neq \hat{\rho}_i$ satisfying

$$\mathbf{E}\left[u_i(\cdot)|(\rho'_i, \hat{\rho}_{-i})\right] > \mathbf{E}\left[u_i(\cdot)|\hat{\rho}\right]. \tag{B.1}$$

Fix this player i throughout the proof.

Note that if one has an incentive to deviate by using a mixed strategy, the player has an incentive to do so by using some pure strategy as well. Assume, therefore, ρ_i' puts probability one on $\zeta_i \neq \zeta_i^*$ in the first round. Let $\omega^* = \lambda(\zeta^*)$ and $\hat{\omega} = \lambda(\zeta_i, \zeta_{-i}^*)$. $\omega^* \neq \hat{\omega}$ holds otherwise i cannot better off. (DA) and strategy profaneness of ζ^* imply $v_i(\omega_i^*) \geq v_i(\hat{\omega}_i)$ and i will trade through a trading cycle to better off:

Let $(k_0, k_1, \ldots, k_{\bar{n}})$ denote this trading cycle. Then, this cycle satisfies the following.

$$k_0 = k_{\bar{n}} = i \tag{B.2}$$

$$v_{k_n}(\hat{\omega}_{k_{n+1}}) > v_{k_n}(\hat{\omega}_{k_n}) \tag{B.3}$$

$$k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n \tag{B.4}$$

Also, note that $k_1, \ldots, k_{\bar{n}}$ are all distinct players otherwise these players does not constitute a trading cycle. In addition, let $O^e = \{\hat{\omega}_{k_1}, \hat{\omega}_{k_2}, \ldots, \hat{\omega}_{k_{\bar{n}}}\}$ be the set of object types exchanged by the players in this trade cycle.

Now, we consider an auxiliary situation by running DA without i at first and then adding i later. Note that DA object allocation is not affected by the order of moves as discussed in Dubins and Freedman (1981)[10]. First, we run DA algorithm without i. After this algorithm is tentatively terminated, we put in player i in the algorithm and continue it until it stops. Let t^* be the step right after the algorithm is tentatively terminated, i.e., at step t^*, i is put in the algorithm. Also, let η be a profile of players' object holdings except i when the algorithm is tentatively terminated. Note that η is stable if we restrict attention to players except for i.

Lemma B.3. *After t^*, suppose i obtains ω_i in step $\bar{t}$ by submitting some ζ_i. Then, i is never accepted at $a_1 \neq \omega_i$ before step $\bar{t}$. Moreover, i will never rejected at ω_i after $\bar{t}$.*

Proof of Lemma B.3

Suppose not, i.e., $\exists t_1 \exists a_1 \neq \omega_i$, i obtains a_1 in $t^* < t_1 < \bar{t}$. In t_1, if a_1 is a leftover, this is a contradiction because i cannot obtain ω_i. Then, there exists j_1 who is rejected at a_1 by i in t_1. Since i obtains ω_i in step $\bar{t}$, there exists a step t s.t. i is rejected at a. Let (a', i', t') be a rejection triple that describes a situation in which $i' \in N$ is at $a' \in O$ and rejected at a' in step t'. Then, we have a rejection chain to push out i from a_1.

$$(a, j', t) = (a_1, j_1, t_1), (a_2, j_2, t_2), \ldots, (a_{\bar{\kappa}}, j_{\bar{\kappa}}, t_{\bar{\kappa}}) = (a, i, t'),$$

where j_κ is rejected at a_κ as $j_{\kappa-1}$ chooses a_κ in step t_κ ($\kappa = 2, \ldots, \bar{\kappa}-1$). Suppose that all the objects except for a in the rejection chain are distinct. Then, this chain of rejection triples constitute a cycle of priority,

$$j_{\bar{\kappa}-1} \succ_a i \succ_a j_1 \succ_{a_2} j_2 \succ_{a_3} \cdots \succ_{a_{\bar{\kappa}-1}} j_{\bar{\kappa}-1}.$$

This is a contradiction.

Next, suppose that all the objects except for a in the rejection chain are not distinct. Then, we can also find a shorter cycle of priority than before by the same argument.

The same argument applies to the case after $\bar{t}$. If i is rejected at ω_i, we can find a priority cycle. This completes the proof of Lemma B.3. □

[10]DA algorithm discussed in this auxiliary situation is essentially the same as the one defined in Dubins and Freedman (1981).

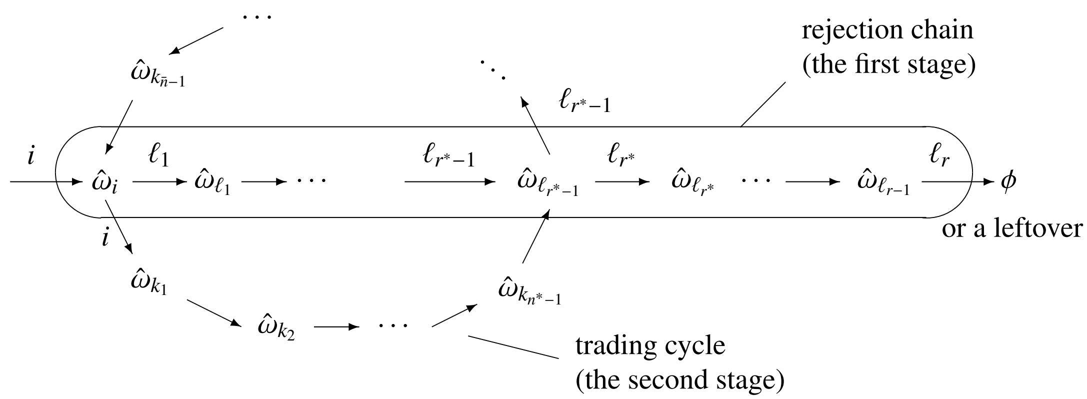

Figure B.1. A Trading Cycle and a Rejection Chain with Intersection

Let us continue the proof of the theorem. By Lemma B.3, i is accepted at $\hat\omega_i$ for the first time under $\hat\zeta_i$.

When i comes to $\hat\omega_i$ at the step $\bar t$, there must be a player, say ℓ_1, in $\hat\omega_i$ otherwise nobody wants to obtain $\hat\omega_i$. And, ℓ_1 is rejected when i comes to $\hat\omega_i$ at step $\bar t$. Let $\bar t = \tau_1$. A rejection triple $(\hat\omega_i, \ell_1, \tau_1)$ denotes the situation in which ℓ_1 is at $\hat\omega_i$ in step $\tau_1 - 1$ and rejected at $\hat\omega_i$ in step τ_1.

Note that after step t^*, DA ends at step τ_{r+1} when player ℓ_r, goes to ϕ or a remaining object after rejected from an object $\hat\omega_{\ell_{r-1}} \in O$. Also, after ℓ_1 is rejected from $\hat\omega_i$ at step τ_1, only one player goes to a new object at every step till τ_{r+1} under the assumptions.

Then, there is a chain of rejection triples $(\hat\omega_{\ell_0}, \ell_1, \tau_1)$, …, $(\hat\omega_{\ell_{r-1}}, \ell_r, \tau_r)$, where $\ell_0 = i$ and for each $r' = 1,\dots,r$, $\ell_{r'}$ is rejected at $\hat\omega_{r'-1}$ at $\tau_{r'}$. Note that $\hat\omega_{r'}$ is the object in O that is obtained by $\ell_{r'}$ in the first stage under $(\sigma_i', \hat\sigma_{-i})$. Note also that all the objects in this chain are distinct; otherwise, there is a cycle of priority. This can be shown by the same procedure as in the proof of Lemma B.3.

There exists an object in O^e that appears in the rejection chain since at least $\hat\omega_i$ is in O^e. Therefore, at least one player, either player i or the one who is rejected after step τ_1, goes to an object in O^e. Let $r^* = 1,\dots,r$ be the greatest number among r's such that $\hat\omega_{\ell_{r'-1}}$ is in O^e, and $(\hat\omega_{\ell_{r'-1}}, \ell_{r'}, \tau_{r'})$ is in the rejection chain. Note that the last player ℓ_r in the chain goes to ϕ or a leftover, which is not in O^e. Therefore, ℓ_{r^*} must go to some object not in O^e. This implies that ℓ_{r^*} is not in the trading cycle. Let n^* be a number such that $k_{n^*} = \ell_{r^*-1}$.

Note that $\eta_{\ell_{r^*}} = \hat\omega_{k_{n^*}}$. Now, consider a player k_{n^*-1} who is in the trading-cycle and will obtain $\hat\omega_{k_{n^*}}$ after the exchange. For the player k_{n^*-1}, $v_{k_{n^*-1}}(\hat\omega_{k_{n^*-1}}) < v_{k_{n^*-1}}(\hat\omega_{k_{n^*}}) = \eta_{\ell_r^*}$ holds: the inequality is implied by the equation B.3. Also, k_{n^*-1} has been rejected at $\hat\omega_{k_{n^*}}$ by step t^* under the truth-telling strategy, otherwise we can find a cycle of priority. Thus, when we consider η, $v_i(\eta_{\ell_{r^*}}) > v_i(\eta_{k_{n^*-1}})$ holds.

Then, the stability of η implies that $\ell_{r^*} \succ_{\hat{\omega}_{k_{n^*}}} k_{n^*-1}$. Also, the rejection chain implies that $k_{n^*} \succ_{\hat{\omega}_{k_{n^*}}} \ell_{r^*}$ holds.

Therefore, we can find a cycle of priority consisting of players in the trading cycle and ℓ_{r^*} rejected from $\hat{\omega}_{k_{n^*}}$ (note here that k_{n^*} is identical with ℓ_{r^*-1}; see Figure B.1),

$$k_{n^*} \succ_{\hat{\omega}_{k_{n^*}}} \ell_{r^*} \succ_{\hat{\omega}_{k_{n^*}}} k_{n^*-1} \cdots \succ_{\hat{\omega}_{k_1}} i \succ_{\hat{\omega}_i} k_{\bar{n}-1} \ldots k_{n^*+1} \succ_{\hat{\omega}_{k_{n^*+1}}} k_{n^*}.$$

Hence, this is a contradiction.

Proof of necessity.

This is a proof by the contraposition. Suppose that the priority has a cycle, i.e., for distinct objects $a, b \in O$ and distinct players $i, j, k \in N$, $i \succ_a j \succ_a k \succ_b i$ holds.

Now, consider A such that for all $\ell \in N \setminus \{i, j, k\}$, $A_\ell = \{\phi\}$ and $A_i = A_j = A_k = \bar{O}$.

Also, consider the following $v \in V_+$[11].

	i	j	k
a	10	30	30
b	30	10	20
c	20	20	10

Values

Given these A and v, SME object allocation is (c, a, b). However, i has an incentive to deviate from the truth-telling strategy and obtain a in the first stage. By deviation, i can always trade a with b in the second stage. There is no PME where i does not obtain b after the market exchange. Thus, this SME object allocation cannot be achieved as a PME object allocation.

□

B.3.2. *(Boston).*

Proof. Proof of sufficiency:

Assume (Quota1), and (Boston). Suppose there exists A, $v \in V_+$, SME object allocation μ is not a PME object allocation.

We want to show that there exists a cycle of priority.

Let $\omega^* = \mu$. Then, there exists a strategy ζ^* such that the players obtain their final objects ω^* and for all $\ell \in N$, $\omega_\ell \in O$ implies that ω_ℓ is at the top of the submitted list.

Since SME object allocation μ is not a PME object allocation, there exists i who gains by deviation from ζ^*. Fix this player i. Let ζ_i be the deviating strategy, and let $\hat{\omega} = \lambda(\zeta_i, \zeta^*_{-i})$. By the construction of ζ^* and stability of ω^*, there exists a player j who is rejected from $\hat{\omega}_i$ by i's deviation. Otherwise, i must have taken a leftover or ϕ. Then, i cannot trade $\hat{\omega}_i$ with an object that is better than ω^*_i. Note that stability of ω^* implies that i cannot acquire an object that is preferable to ω^*_i in the first stage by any deviation.

[11] Although genericity is violated in this v, the argument does not depend upon non-genericity.

Thus, i has an incentive to deviate only when there is a trading cycle in the second stage due to i's deviation. Let $(k_0, k_1, \dots, k_{\bar{n}})$ denote this trading cycle. Then, this cycle satisfies the following.

$$k_0 = k_{\bar{n}} = i \tag{B.5}$$

$$v_{k_n}(\hat{\omega}_{k_{n+1}}) > v_{k_n}(\hat{\omega}_{k_n}) \tag{B.6}$$

$$k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n \tag{B.7}$$

Also, trading cycle must be consisted of distinct players.

Next, we show the following claim.

Claim: j is not in the trading cycle, i.e., $j \neq k_0, \dots, k_{\bar{n}}$.

After rejected from $\hat{\omega}_i$, an object j may obtain in the first stage is ω_i^* or another remaining object or ϕ. If j obtains a a remaining object or ϕ, j is not in a trading cycle as nobody is interested in the leftover under SME.

Then, this claim implies that there exists a following generalized cycle of priority.

$i \succ_{\hat{\omega}_i} j \succ_{\hat{\omega}_i} k_{\bar{n}-1} \succ_{\hat{\omega}_{k_{\bar{n}-1}}} \dots \succ_{\hat{\omega}_{k_2}} k_1 \succ_{\hat{\omega}_{k_1}} i.$

This is a contradiction.

Note that although we have considered a specific strategy ζ^*, this proof does not loose the generality. Whenever a SME object allocation is a PME object allocation, we can construct a strategy as described in ζ^* and find a generalized priority cycle.

Therefore, this ends the proof of sufficiency. Proof of necessity is same in the case of (DA).

□

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