Two Extensions: Unobserved Heterogeneity and Finite Dependence

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Introduction

Moving away from the benchmark

- We have already shown that if no restrictions are placed on how the observed variables affect utility: the model is exactly identified off a long panel if one of the payoffs is known for each state and if the distribution of unobserved variables is known.
- Therefore, absent further restrictions, the model loses point identification if:
    1. the panel is short rather than long
    2. the distribution of unobserved variables is unknown
    3. the discount factor is unknown
    4. the payoff from one choice for each state is unknown
- In empirical applications we typically parameterize the effects of observed variables, freeing up equations to relax these restrictions.
- This lecture is a preliminary investigation of the first two issues.

- Adam Smith, and many others, including perhaps your parents, have commented on "the hasty, fond, and foolish intimacies of young people" (Smith, page 395, volume 1, 1812).
- One approach to explaining such behavior is to argue that some people are not rational all the time.
- A challenge for this approach is to develop an axiomatic theory for irrational agents that has refutable predictions.
- There is ongoing research in behavioral economics and economic theory in this direction.
- Another approach, embraced by many labor economists, is that by repeatedly sampling experiences from an unfamiliar environment, rational Bayesians update their prior beliefs as they sequentially solve their lifecycle problem.

- This issue seems like a candidate for applying the methodology described in the previous slides:
  1. Write down a dynamic discrete choice model of Bayesian updating and sequential optimization problem;
  2. Solve the individual's optimization problem (for all possible parameterizations of the primitives);
  3. Treat important factors to the decision maker that are not reported in the sample population as unobserved variables to the econometrician;
  4. Integrating over the probability distribution of unobserved random variables, form the likelihood of observing the sample;
  5. Maximize the likelihood to obtain the structural parameters that characterize the dynamic discrete choice problem;
  6. Predict how the individual would adjust her behavior if she was confronted with new opportunities to learn or different payoffs.

Job Matching and Occupational Choice (Miller JPE, 1984)

Individual payoffs and choices

- The payoff from job $m \in M$ at time $t \in \{0, 1, \ldots\}$ is:

$$x_{mt} \equiv \psi_t + \xi_m + \sigma_m \epsilon_{mt}$$

  where:
  - ψ_t is a lifecycle trend shaping term that plays no role in the analysis;
  - ξ_m is a job match parameter drawn from $N\left(\gamma_m, \delta_m^2\right)$;
  - ε_{mt} is an idiosyncratic *iid* disturbance drawn from $N\left(0, 1\right)$
- Every period t the individual chooses a job m to work in. The choice at t is denoted by $d_{mt} \in \{0, 1\}$ for each $m \in M$ where:

$$\sum_{m \in M} d_{mt} = 1$$

- The realized lifetime utility of the individual is:

$$\sum_{t=0}^{\infty} \sum_{m \in M} \beta^t d_{mt} x_{mt}$$

Job Matching and Occupational Choice

Processing information

- At $t = 0$ the individual sees $\left(\gamma_m, \delta_m^2\right)$ for all $m \in M$.
- At every t, after making her choice, she also sees ψ_t, and $d_{mt} x_{mt}$ for all $m \in M$.
- Following Degroot *(Optimal Statistical Decisions 1970, McGraw Hill)* the posterior beliefs of an individual for job $m \in M$ at time $t \in \{0, 1, \ldots\}$ are $N\left(\gamma_{mt}, \delta_{mt}^2\right)$ where:

$$
\begin{aligned}
\gamma_{mt} &= \frac{\delta_m^{-2} \gamma_m + \sigma_m^{-2} \sum_{s=0}^{t-1} \left(x_{ms} - \psi_s\right) d_{ms}}{\delta_m^{-2} + \sigma_m^{-2} \sum_{s=0}^{t-1} d_{ms}} \\
\delta_{mt}^{-2} &= \delta_m^{-2} + \sigma_m^{-2} \sum_{s=0}^{t-1} d_{ms}
\end{aligned}
$$

- She maximizes the sum of expected payoffs, sequentially choosing d_{mt} for each $m \in M$ at t given her beliefs $N\left(\gamma_{mt}, \delta_{mt}^2\right)$.

Corollary (from Theorem 2 in Gittens and Jones,1974)

At each $t \in \{1, 2, \ldots\}$ it is optimal to select the $m \in M$ maximizing:

$$DAI_m(\gamma_{mt}, \delta_{mt}) \equiv \sup_{\tau \geq t} \left\{ \frac{E\left[\sum_{r=t}^{\tau} \beta^r (x_{mr} - \psi_r) \,|\gamma_{mt}, \delta_{mt}\right]}{E\left[\sum_{r=t}^{\tau} \beta^r \,|\gamma_{mt}, \delta_{mt}\right]} \right\}$$

- To understand the intuition for this rule, consider two projects, m' taking 4 periods with payoffs $\{1, 8, 7, x'\}$ and another m'' taking 2 periods with payoffs $\{6, x''\}$.
- Suppose m' can be split into a 3 period project with payoffs $\{1, 8, 7\}$ and an additional 1 period project with payoff $\{x'\}$ that cannot be undertaken before the 3 period project is completed, but does not have to be undertaken immediately afterwards.
- Prove the DAI rule optimally schedules the projects.

- Proposition 4 of Miller (1984) shows:

$$DAI_m\left(\gamma_{mt},\delta_{mt}\right)=\gamma_{mt}+\delta_{mt}D\left[\left(\frac{\sigma_m}{\delta_m}\right)^2+\sum\nolimits_{s=0}^{t-1}d_{ms}\right]$$

  where $D\left(\cdot\right)$ is the (standard) DAI for a (hypothetical) job whose match parameter ξ is drawn from $N\left(0,1\right)$ and whose payoff net of the general component is $\sigma^2\varepsilon_t$.
- $D\left(\cdot\right)$ can be numerically computed by solving for the fixed point of a contraction mapping. (See Proposition 5 of Miller, 1984.):$D\left(\cdot\right)$ is a deceasing function. Thus $DAI_m\left(\gamma_{mt},\delta_{mt}\right)\uparrow$ as:
  - γ_{mt}, δ_{mt} and $\delta_m\uparrow$
  - σ_m and $\sum_{s=0}^{t-1}d_{ms}\downarrow$.
- Given γ_m:
  - Occupations with high δ_m and low σ_m are experimented with first;
  - Matches with low σ_m are resolved for better or worse relatively quickly;
  - Turnover declines with tenure.

- Define h_t as the (discrete) hazard at t periods as the probability a spell ends after t periods conditional on surviving that long.
- In a one occupation mode and only keep track of the current job match. (Why?)

$$\begin{aligned} h_t &\equiv \Pr\left\{\gamma_t + \delta_t D\left[\left(\frac{\sigma}{\delta}\right)^2 + t, \beta\right] \leq \gamma + \delta D\left[\left(\frac{\sigma}{\delta}\right)^2, \beta\right]\right\} \\ &= \Pr\left\{\frac{\gamma_t - \gamma}{\sigma} \leq \frac{\delta}{\sigma} D\left[\left(\frac{\sigma}{\delta}\right)^2, \beta\right] - \frac{\delta_t}{\sigma} D\left[\left(\frac{\sigma}{\delta}\right)^2 + t, \beta\right]\right\} \\ &= \Pr\left\{\rho_t \leq \alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right\} \end{aligned}$$

where $\rho_t \equiv (\gamma_t - \gamma)/\sigma$ and $\alpha \equiv \sigma/\delta$ which implies:

$$\frac{\delta_t}{\sigma} = \frac{\left[\delta^{-2} + t\sigma^{-2}\right]^{-1/2}}{\sigma} = \left[\left(\frac{\delta}{\sigma}\right)^{-2} + t\right]^{-1/2} = (\alpha + t)^{-1/2}$$

Probability Distribution of Spell Lengths

Relating the hazard rate to the distribution of normalized match qualities

- Define the probability distribution of transformed means of spells surviving at least t periods as:

$$\Psi_t(\rho) \equiv \Pr\{\rho_t \leq \rho\} = \Pr\left\{\sigma^{-1}(\gamma_t - \gamma) \leq \rho\right\} = \Pr\{\gamma_t \leq \gamma + \rho\sigma\}$$

- To help fix ideas note that $\Psi_0(\rho) = 0$ for all $\rho \leq 0$ and $\Psi_0(0) = 1$.
- From the definition of h_t and $\Psi_t(\rho)$:

$$\begin{aligned} h_t &= \Pr\left\{\rho_t \leq \alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right\} \\ &= \Psi_t\left[\alpha^{-1/2} D(\alpha, \beta) - (\alpha + t)^{-1/2} D(\alpha + t, \beta)\right] \end{aligned}$$

- To derive the discrete hazard, we recursively compute $\Psi_t(\rho)$.

Probability Distribution of Spell Lengths

Inequalities relating to normalized match qualities after one period

- By definition every match survives at least one period, and hence:

$$\Psi_1(\rho) = \Pr\{\gamma_1 \leq \gamma + \rho\sigma\}$$

- From the Bayesian updating rule for γ_t:

$$\begin{aligned}
\gamma_1 &\leq \gamma + \rho\sigma \\
&\Leftrightarrow \frac{\delta^{-2}\gamma + \sigma^{-2}(x_1 - \psi_1)}{\delta^{-2} + \sigma^{-2}} \leq \gamma + \rho\sigma \\
&\Leftrightarrow \delta^{-2}\gamma + \sigma^{-2}(\xi + \sigma\epsilon) \leq (\gamma + \rho\sigma)\left(\delta^{-2} + \sigma^{-2}\right) \\
&\Leftrightarrow \alpha\gamma + \xi + \sigma\epsilon \leq (\gamma + \rho\sigma)(\alpha + 1) \\
&\Leftrightarrow (\xi - \gamma) + \sigma\epsilon \leq \sigma(\alpha + 1)\rho \\
&\Leftrightarrow \delta^{-1}(\xi - \gamma) + \alpha^{1/2}\epsilon \leq \alpha^{1/2}(\alpha + 1)\rho
\end{aligned}$$

Probability Distribution of Spell Lengths

Computing the distribution of normalized match qualities after one period

- By definition every match survives at least one period, and hence:

$$\Psi_1(\rho) \equiv \Pr\{\gamma_1 \leq \gamma + \rho\sigma\}$$

- Appealing to the inequalities from the previous slide:

$$\begin{aligned}
\Psi_1(\rho) &= \Pr\{\gamma_1 \leq \gamma + \rho\sigma\} \\
&= \Pr\left\{\delta^{-1}(\xi - \gamma) + \alpha^{1/2}\epsilon \leq \alpha^{1/2}(\alpha+1)\rho\right\} \\
&= \Pr\left\{\epsilon' + \alpha^{1/2}\epsilon \leq \alpha^{1/2}(\alpha+1)\rho\right\} \\
&= \Pr\left\{(\alpha+1)^{1/2}\epsilon'' \leq \alpha^{1/2}(\alpha+1)\rho\right\} \\
&= \Phi\left[\alpha^{1/2}(\alpha+1)^{1/2}\rho\right]
\end{aligned}$$

where ϵ' and ϵ'' are random variables both distributed independently as standard normal.

Probability Distribution of Spell Lengths

Solving for the one period hazard rate and the probability distribution of survivors

- The spell ends if:

$$\rho_1 < \alpha^{-1/2} D(\alpha, \beta) - (\alpha + 1)^{-1/2} D(\alpha + 1, \beta)$$

- Therefore the proportion of spells ending after one period is:

$$\begin{aligned} h_1 &= \Psi_1 \left[\alpha^{-1/2} D(\alpha, \beta) - (\alpha + 1)^{-1/2} D(\alpha + 1, \beta) \right] \\ &= \Phi \left\{ \begin{array}{l} \left[\alpha^{1/2} (\alpha + 1)^{1/2} \right] \\ \times \left[\alpha^{-1/2} D(\alpha, \beta) - (\alpha + 1)^{-1/2} D(\alpha + 1, \beta) \right] \end{array} \right\} \\ &> 1/2 \end{aligned}$$

- So the truncated distribution of ρ for survivors after one draw is:

$$\widetilde{\Psi}_1(\rho) \equiv (1 - h_1)^{-1} \left[\Psi_1(\rho) - h_1 \right]$$

Probability Distribution of Spell Lengths

Recursively computing the distribution of normalized match qualities

- To derive $\Psi_2(\rho)$ from $\widetilde{\Psi}_1(\rho)$ the worker takes another draw, and appealing to Bayes rule one more time:

$$\begin{aligned}\Psi_2(\rho) &\equiv \frac{\int_{-\infty}^{\infty} \Psi_1\left(\rho - \epsilon\left[(\alpha+1)(\alpha+2)\right]^{-1/2}\right) d\Phi(\epsilon) - h_1}{1-h_1} \\ &= \frac{\int_{-\infty}^{\infty} \Phi\left[\begin{array}{l} \alpha^{1/2}(\alpha+1)^{1/2} \times \\ \left(\rho - \epsilon\left[(\alpha+1)(\alpha+2)\right]^{-1/2}\right) \end{array}\right] d\Phi(\epsilon) - h_1}{1-h_1}\end{aligned}$$

- More generally (from page 1112 of Miller, 1984):

$$\Psi_{t+1}(\rho) \equiv \frac{\int_{-\infty}^{\infty} \Psi_t\left(\rho - \epsilon\left[(\alpha+t)(\alpha+t+1)\right]^{-1/2}\right) d\Phi(\epsilon) - h_t}{1-h_t}$$

- Suppose the sample comprises a cross section of spells $n \in \{1, \ldots, N\}$, some of which are completed after τ_n periods, and some of which are incomplete lasting at least τ_n periods. Let:

$$\rho(n) \equiv \begin{cases} \tau_n \text{ if spell is complete} \\ \{\tau_n, \tau_{n+1}, \ldots\} \text{ if spell is incomplete} \end{cases}$$

- Let $p_\tau(\alpha_n, \beta_n)$ denote the unconditional probability of individual n with discount factor β_n working τ periods in a new job with information factor α_n before switching to another new job in the same occupation:

$$p_\tau(\alpha_n, \beta_n) \equiv h_\tau(\alpha_n, \beta_n) \prod_{s=1}^{\tau-1} [1 - h_s(\alpha_n, \beta_n)]$$

- Then the joint probability of spell duration times observed in the sample is:

$$\prod_{n=1}^{N} \sum_{\tau \in \rho(n)} p_\tau(\alpha_n, \beta_n)$$

Maximum Likelihood Estimation

The likelihood function and structural estimates

- We could allow for an additional source of unobserved heterogeneity by writing the likelihood as:

$$L_N(A_1, B_1, A_2, B_2, \lambda) \equiv \prod_{n=1}^{N} \sum_{\tau \in \rho(n)} \left[\begin{array}{l} p_\tau(\alpha_{1n}, \beta_{1n})\lambda \\ +p_\tau(\alpha_{1n}, \beta_{1n})(1-\lambda) \end{array} \right]$$

  where we now assume that $\alpha_{in} \equiv A_i X_n$ and $\beta_n \equiv B_i X_n$ for $i \in \{1, 2\}$ and the parameter space is $(A_1, B_1, A_2, B_2, \lambda)$.

- Briefly, the structural estimates show that:
  1. individuals care about the future and value on job experimentation;
  2. the occupational dummy variables are significant, suggesting that the choice of different occupations is not random;
  3. educational groups have different beliefs and learning rates;
  4. these three results are not sensitive to whether the additional unobserved heterogeneity is incorporated or not.

- Is there an easier way?
- An alternative approach is to combine the inversion theorem, to avoid solving the dynamic optimization (equilibrium) problem, with a variation on the EM (Expectation/Maximization) algorithm, to handle the unobserved heterogeneity.
- Recall Mr. Zurcher decides whether to replace the existing engine ($d_{1t} = 1$), or keep it for at least one more period ($d_{2t} = 1$).
- Bus mileage advances 1 unit ($x_{t+1} = x_t + 1$) if Zurcher keeps the engine ($d_{2t} = 1$) and is set to zero otherwise($x_{t+1} = 0$ if $d_{1t} = 1$).
- Transitory iid choice-specific shocks, ϵ_{jt} are Type 1 Extreme value.
- Zurcher sequentially maximizes expected discounted sum of payoffs:

$$E\left\{\sum_{t=1}^{\infty}\beta^{t-1}\left[d_{2t}(\theta_1 x_t + \theta_2 s + \epsilon_{2t}) + d_{1t}\epsilon_{1t}\right]\right\}$$

Motivating Example

ML Estimation when CCP's are known (infeasible)

- To show how the EM algorithm helps, consider the infeasible case where $s \in \{1, \ldots, S\}$ is unobserved but $p(x, s)$ is known.
- Let π_s denote population probability of being in unobserved state s.
- Supposing β is known the ML estimator for this "easier" problem is:

$$\{\hat{\theta}, \hat{\pi}\} = \arg\max_{\theta,\pi} \sum_{n=1}^{N} \ln \left[\sum_{s=1}^{S} \pi_s \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s, p, \theta) \right]$$

  where $l(d_{nt}|x_{nt}, s_n, p, \theta)$ takes the form:

$$\frac{d_{1nt} + d_{2nt} \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln[p(0,s)] - \beta \ln[p(x_{nt}+1,s)]}{1 + \exp(\theta_1 x_{nt} + \theta_2 s + \beta \ln[p(0,s)] - \beta \ln[p(x_{nt}+1,s)])}$$

  and $p \equiv p(x, s)$ is the string of (assumed) known (x, s).
- Maximizing over the sum of a log of summed products is computationally burdensome.

- The EM algorithm is a computationally attractive alternative to directly maximizing the likelihood.
- Denote by $d_n \equiv (d_{n1}, \ldots, d_{nT})$ and $x_n \equiv (x_{n1}, \ldots, x_{nT})$ the full sequence of choices and mileages observed in the data for bus n.
- At the m^{th} iteration:

$$
\begin{aligned}
q_{ns}^{(m+1)} &= \Pr\left\{ s \,\middle|\, d_n, x_{n,} \theta^{(m)}, \pi_s^{(m)}, p \right\} \\
&= \frac{\pi_s^{(m)} \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s, p, \theta^{(m)})}{\sum_{s'=1}^{S} \pi_{s'}^{(m)} \prod_{t=1}^{T} l(d_{nt}|x_{nt}, s', p, \theta^{(m)})}
\end{aligned}
$$

$$
\pi_s^{(m+1)} = N^{-1} \sum_{n=1}^{N} q_{ns}^{(m+1)}
$$

$$
\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln[l(d_{nt}|x_{nt}, s, p, \theta)]
$$

Motivating Example

Steps in our algorithm when s is unobserved and CCP's are unknown

Our algorithm begins by setting initial values for $\theta^{(1)}$, $\pi^{(1)}$, and $p^{(1)}(\cdot)$:

Step 1 Compute $q_{ns}^{(m+1)}$ as:

$$q_{ns}^{(m+1)} = \frac{\pi_s^{(m)} \prod_{t=1}^{T} l\left[d_{nt} | x_{nt}, s, p^{(m)}, \theta^{(m)}\right]}{\sum_{s'=1}^{S} \pi_s^{(m)} \prod_{t=1}^{T} l\left(d_{nt} | x_{nt}, s', p^{(m)}, \theta^{(m)}\right)}$$

Step 2 Compute $\pi_s^{(m+1)}$ according to:

$$\pi_s^{(m+1)} = \frac{\sum_{n=1}^{N} q_{ns}^{(m+1)}}{N}$$

Step 3 Update $p^{(m+1)}(x, s)$ using one of two rules below

Step 4 Obtain $\theta^{(m+1)}$ from:

$$\theta^{(m+1)} = \arg\max_{\theta} \sum_{n=1}^{N} \sum_{s=1}^{S} \sum_{t=1}^{T} q_{ns}^{(m+1)} \ln\left[l\left(d_{nt} | x_{nt}, s_n, p^{(m+1)}, \theta\right)\right.$$

- Take a weighted average of decisions to replace engine, conditional on x, where weights are the conditional probabilities of being in unobserved state s.

  Step 3A Update CCP's with:

$$p^{(m+1)}(x, s) = \frac{\sum_{n=1}^{N} \sum_{t=1}^{T} d_{1nt} q_{ns}^{(m+1)} I(x_{nt} = x)}{\sum_{n=1}^{N} \sum_{t=1}^{T} q_{ns}^{(m+1)} I(x_{nt} = x)}$$

- Or in a stationary infinite horizon model use identity from model that likelihood returns CCP of replacing the engine:

  Step 3B Update CCP's with:

$$p^{(m+1)}(x_{nt}, s_n) = l(d_{nt1} = 1 | x_{nt}, s_n, p^{(m)}, \theta^{(m)})$$

First Monte Carlo

Finite horizon renewal problem

- Suppose $s \in \{0, 1\}$ equally weighted.
- There are two observed state variables
  1. total accumulated mileage:

$$x_{1t+1} = \begin{cases} \Delta_t \text{ if } d_{1t} = 1 \\ x_{1t} + \Delta_t \text{ if } d_{2t} = 1 \end{cases}$$

  2. permanent route characteristic for the bus, x_2, that systematically affects miles added each period.
- We assume $\Delta_t \in \{0, 0.125, \ldots, 24.875, 25\}$ is drawn from:

$$f(\Delta_t | x_2) = \exp\left[-x_2(\Delta_t - 25)\right] - \exp\left[-x_2(\Delta_t - 24.875)\right]$$

and x_2 is a multiple 0.01 drawn from a discrete equi-probability distribution between 0.25 and 1.25.

- Let θ_{0t} be an aggregate shock (denoting cost fluctuations say).
- The difference in current payoff from retaining versus replacing the engine is:

$$u_{2t}(x_{1t}, s) - u_{1t}(x_{1t}, s) \equiv \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s$$

- Denoting the observed state variables by $x_t \equiv (x_{1t}, x_2)$, this translates to:

$$\begin{aligned} v_{2t}(x_t, s) - v_{1t}(x_t, s) &= \theta_{0t} + \theta_1 \min\{x_{1t}, 25\} + \theta_2 s \\ &\quad + \beta \sum_{\Delta_t \in \Lambda} \left\{ \ln\left[\frac{p_{1t}(0, s)}{p_{1t}(x_{1t} + \Delta_t, s)}\right]\right\} f(\Delta_t | x_2) \end{aligned}$$

First Monte Carlo

Table 1 of Arcidiacono and Miller (2011)

Second Monte Carlo

Structure

- Entrants pay startup cost to compete in the market, but not incumbents.
- Paying startup cost now transforms entrant into incumbent next period.
- Declining to compete in any given period is tantamount to exit.
- When a firm exits another firm potentially enters next period.

Second Monte Carlo

Dynamics

- There are two sources of dynamics in this model.
- An entrant depreciates startup cost over its anticipated lifetime.
- Since it is more costly for an entrant to start operations, than for an incumbent to continue, the number of incumbents signals how much competition the firm faces in the current period, and consequently affects its own decision whether to exit the industry or not.

- Each market has a permanent market characteristic, denoted by x_1, common to each player within the market and constant over time, but differing independently across markets, with equal probabilities on support $\{1, \ldots, 10\}$.
- The number of firm exits in the previous period is also common knowledge to the market, and this variable is indicated by:

$$x_{2t} \equiv \sum_{h=1}^{I} d_{1,t-1}^{(h)}$$

- This variable is a useful predictor for the number of firms that will compete in the current period.
- Intuitively, the more players paying entry costs, the lower the expected number of competitors.

Second Monte Carlo

Unobserved (Markov chain state) variables, and price equation

- The unobserved state variable $s_t \in \{1, \ldots, 5\}$ follows a first order Markov chain.
- We assume that the probability of the unobserved variable remaining unchanged in successive periods is fixed at some $\pi \in (0, 1)$, and that if the state does change, any other state is equally likely to occur with probability $(1 - \pi)/4$.
- We generated also price data on each market, denoted by w_t, with the equation:

$$w_t = \alpha_0 + \alpha_1 x + \alpha_2 s_t + \alpha_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \eta_t$$

where η_t is distributed as a standard normal disturbance independently across markets and periods, revealed to each market after the entry and exit decisions are made.

- The flow payoff of an active firm i in period t, net of private information $\epsilon_{2t}^{(i)}$ is modeled as:

$$U_2\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = \theta_0 + \theta_1 x + \theta_2 s_t + \theta_3 \sum_{h=1}^{I} d_{1t}^{(h)} + \theta_4 d_{1,t-1}^{(i)}$$

- We normalize exit utility as $U_1\left(x_t^{(i)}, s_t^{(i)}, d_t^{(-i)}\right) = 0$
- We assume $\epsilon_{jt}^{(i)}$ is distributed as Type 1 Extreme Value.
- The number of firms in each market in our experiment is 6.
- We simulated data for $3,000$ markets, and set $\beta = 0.9$.
- Starting at an initial date with 6 entrants in the market, we ran the simulations forward for twenty periods.

Second Monte Carlo

Table 2 of Arcidiacono and Miller (2011)

Defining Finite Dependence

Motivation

- The property of finite dependence is motivated by two concerns, one theoretical and one practical.
- Theoretically our data may come from a short panel, where the horizon of the agent extends beyond the length of the panel.
- In practice the identifying equations maybe somewhat unwieldy:
    1. In stationary Markov models identification is achieved by inverting a matrix of dimension X, the number of states;
    2. In finite horizon problems the equations telescope the CCPs out to the last period T of the agent's life.
- Simulation methods alleviate these problems, but Monte Carlo integration is only a way of numerically approximating the integration/inversion.
- Can assumptions on the model eliminate this problem?
- Such assumptions would be placed on the transition matrix, and would therefore be testable.

Defining Finite Dependence

Weighted distribution of state variables induced by weighted choices

- Let $\omega_{jkt\tau}(x, x_\tau)$ denote the weight on the k^{th} action at period $\tau \in \{t+1, t+2, \ldots\}$ when the state is x_τ, was x at t, and action j was taken at t.
- We assume $\omega_{jkt\tau}(x, x_\tau)$ can be positive or negative, but require:

$$\sum_{k=1}^{J} \omega_{jkt\tau}(x, x_\tau) = 1$$

- Recursively define a weight distribution by setting $\kappa_{jt0}(x_{t+1}|x) \equiv f_{jt}(x_{t+1}|x)$, and:

$$\kappa_{jt,\tau-t}(x_{\tau+1}|x) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} f_{k\tau}(x_{\tau+1}|x_\tau)\omega_{jkt\tau}(x, x_\tau)\,\kappa_{jt,\tau-t-1}(x_\tau|x)$$

- When $\omega_{jkt\tau}(x, x_\tau) \geq 0$ for all (k, τ, x_τ) for given (t, x, j) we can interpret ω as a (nonoptimal) randomized decision rule, and $\kappa_{jt,\tau-t-1}(x_\tau|x)$ as the probability of reaching x_τ in period τ from (t, x) by taking choice j at t and then applying ω.

Defining Finite Dependence

Equalizing the weight distribution of state variables for a pair of paths

- Consider two sequences of decision weights beginning at date t in state x, one with choice i and the other with choice j.
- We say that the pair of choices (i, j) exhibits ρ-period dependence at (t, x) if there exists an ω from i and j for x such that for all $x_{t+\rho+1}$:

$$\kappa_{it\rho}(x_{t+\rho+1}|x) = \kappa_{jt\rho}(x_{t+\rho+1}|x) \tag{1}$$

- That is, the weights associated with each state are equalized across the two paths after ρ periods.
- Finite dependence trivially holds in all finite horizon problems, but ρ-period dependence only merits attention when $\rho < T - t$.
- For this reason we ignore the trivial case of $\rho = T - t$.
- Finite dependence is defined for a given player, say n, in an exactly analogous manner, that is for a given date t in state x, and two choices i and j.

Why Finite Dependence Matters

An expression for differences in current payoffs

- Adapting the proof of the representation theorem:

$$\begin{aligned} v_{jt}(x) - u_{jt}(x) &= \sum_{\tau=t+1}^{t+\rho} \sum_{(k,x_\tau)}^{(J,X)} \beta^{\tau-t} \left\{ \begin{array}{l} \left[u_{k\tau}(x_\tau) + \psi_{k\tau}(x_\tau)\right] \\ \times \omega_{jkt\tau}(x, x_\tau) \kappa_{jt,\tau-t-1}(x_\tau|x) \end{array} \right\} \\ &\quad + \sum_{x_{t+\rho+1}}^{X} \beta^{t+\rho+1-t} V_{t+\rho+1}(x_{t+\rho+1}) \kappa_{jt\rho}(x_{t+\rho+1}|x) \end{aligned}$$

- If ρ-period dependence holds at (i, j, t, x) then for some ω:

$$\kappa_{t+\rho}(x_{t+\rho+1}|x, i) = \kappa_{t+\rho}(x_{t+\rho+1}|x, j)$$

- Differencing the expression above with respect to i and j:

$$\begin{aligned} & u_{jt}(x) - u_{it}(x) - \psi_{jt}(x) + \psi_{it}(x) \\ = & \sum_{\tau=t+1}^{t+\rho} \sum_{(k,x_\tau)}^{(J,X)} \beta^{\tau-t} \left\{ \begin{array}{l} \left[u_{k\tau}(x_\tau) + \psi_{k\tau}(x_\tau)\right] \times \\ \left[\begin{array}{l} \omega_{ikt\tau}(x, x_\tau) \kappa_{it,\tau-t-1}(x_\tau|x) \\ -\omega_{jkt\tau}(x, x_\tau) \kappa_{jt,\tau-t-1}(x_\tau|x) \end{array} \right] \end{array} \right\} \end{aligned}$$

- A terminal choice ends the evolution of the state variable with an absorbing state that is independent of the current state.
- That is $f_{1t}(x_{t+1}|x) \equiv f_{1t}(x_{t+1})$ for all (t, x).
- Let the first choice denote a terminal choice. Then:

$$\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{jt}(x_{t+1}|x_t) = f_{1,t+1}(x_{t+2})$$

- From the representation theorem:

$$\begin{aligned} & u_1(x_t) - u_j(x_t) - \psi_{1t}(x) + \psi_{jt}(x) \\ = & \sum_{x_{t+1}=1}^{X} \beta \left[u_1(x) + \psi_{1,t+1}(x) \right] f_{jt}(x|x_t) \end{aligned}$$

- If there is more than one period of data, and $f_{jt}(x|x_t)$ varies with t, then $u_j(x_t)$ is typically (over) identified for all $j \in \{1, \ldots, J\}$.

Simple Examples of Finite Dependence

Renewal choices and stable utility

- Similarly a renewal choice yields a probability distribution of the state variable next period that does not depend on the current state.
- Letting the first choice denote a renewal choice:

$$\begin{aligned}\sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}|x_{t+1}) f_{jt}(x_{t+1}|x_t) &\equiv \sum_{x_{t+1}=1}^{X} f_{1,t+1}(x_{t+2}) f_{1t}(x_{t+1}|x_t) \\ &= f_{1,t+1}(x_{t+2})\end{aligned}$$

- From the representation theorem:

$$\begin{aligned}& u_j(x) - u_1(x) - \psi_{jt}(x) + \psi_{1t}(x) \\ = & \sum_{x'=1}^{X} \beta \left[u_1(x') + \psi_{1,t+1}(x')\right] \left[f_{1t}(x'|x) - f_{jt}(x'|x)\right]\end{aligned}$$

- Notice that we obtain T equations for each x, which can be used in identification.

Estimation with Finite Dependence

The data

- Suppose there are N observations of the state variables and decisions denoted by $\{d_{nt_n}, x_{nt_n}, x_{n,t_n+1}\}_{n=1}^{N}$ sampled within a time frame of $t \in \{1, \ldots, \mathcal{T}\}$.
- Say there are M separate instances of finite dependence as defined in (1) within that time frame where, for the sake of exposition, each pair of choices includes choice 1.
- Label the M paths by (j_m, x_m, t_m, ρ_m) for $m \in \{1, \ldots, M\}$.
- Assume that for each $t \in \{1, \ldots, \mathcal{T}\}$ the probability of the sample selection mechanism drawing $x \in \{1, \ldots, X\}$ is strictly positive.

Estimation with Finite Dependence

Standard assumptions

- Assume the subjective discount factor β, and $g(\epsilon_t)$, the joint probability density function for the unobserved idiosyncratic taste shock ϵ_t, are known.
- Second, assume $u_{jt}(x)$ can be parameterized by a finite dimensional vector $\theta \equiv (\theta_1, \ldots, \theta_K) \in \Theta$, a closed convex set in $\mathcal{R}^K$.
- Normalize the first choice to zero, by writing $u_{jt}(x) = \widetilde{u}_{jt}(x, \theta)$, where $\widetilde{u}_{jt}(x, \theta)$ is a known function with $\widetilde{u}_{0t}(x, \theta) = 0$ for all (t, x).
- Finally, assume that the M instances of finite dependence are sufficient to identify θ.

Estimation with Finite Dependence

First stage: unrestricted CCP and transition probabilities

- For all $t \in \{1, \ldots, \mathcal{T}\}$ and $x \in \{1, \ldots, X\}$, define the cell estimators of $p_{jt}(x)$ as:

$$\widehat{p}_{jt}(x) \equiv \frac{\sum_{n=1}^{N} 1\left\{d_{nt(n)j} = 1\right\} 1\left\{t_n = t\right\} 1\left\{x_{nt(n)} = x\right\}}{\sum_{n=1}^{N} 1\left\{t_n = t\right\} 1\left\{x_{nt(n)} = x\right\}}$$

  and estimate the $XJ\mathcal{T}$ CCP vector $p \equiv (p_{11}(1), \ldots, p_{J\mathcal{T}}(X))'$ with $\widehat{p}$ formed from $\widehat{p}_{jt}(x)$.

- If the state transitions are unknown, estimate $f_{jt}(x)$ with $\widehat{f}_{jt}(x)$ in this first stage, for example with a cell estimator (similar to the CCP estimator).

- For any is an M dimensional positive definite matrix W:

$$\widehat{\theta} \equiv \arg\min_{\theta} \left[y\left(\widehat{p}, \widehat{f}\right) - Z\left(\widehat{p}, \widehat{f}, \theta\right) \right]' W \left[y\left(\widehat{p}\right) - Z\left(\widehat{p}, \widehat{f}, \theta\right) \right] \quad (2)$$

where $y\left(p, f\right)$ is an M dimensional vector with elements $y_m\left(p, f\right)$ and $Z(p, f, \theta)$ is an M dimensional vector with elements $Z_m(p, f, \theta)$, with:

$$y_m\left(p, f\right) \equiv \psi_1[p_{t(m)}(x_m)] - \psi_{j(m)}[p_{t(m)}(x_m)]$$
$$+ \sum_{\tau=t_m+1}^{t_m+\rho_m} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta^{\tau-t(m)} \psi_k[p_\tau(x_\tau)] \left[\begin{array}{l} \omega_{k\tau}(x_\tau, 1)\kappa_\tau(x_\tau|x_m, 1) - \\ \omega_{k\tau}(x_\tau, j_m)\kappa_\tau(x_\tau|x_m, j_m) \end{array} \right]$$

$$Z_m(p, f, \theta) \equiv \widetilde{u}_{j(m),t(m)}(x_m, \theta)$$
$$- \sum_{\tau=t_m+1}^{t_m+\rho_m} \sum_{k=1}^{J} \sum_{x_\tau=1}^{X} \beta_{k\tau}^{\tau-t(m)} \widetilde{u}_{k\tau}(x_\tau, \theta) \left[\begin{array}{l} \omega_{k\tau}(x_\tau, 1)\kappa_\tau(x_\tau|x_m, 1) - \\ \omega_{k\tau}(x_\tau, j_h)\kappa_\tau(x_\tau|x_m, j_m) \end{array} \right]$$

- At the true parameter values $y_t\left(x,p,f\right)=Z_t(x,p,f,\theta)$.
- Hence $\widehat{\theta}$ is $\sqrt{N}$ consistent and asymptotically normal.
- Setting $W=\widehat{W}$, a consistent estimate of the inverse of the asymptotic covariance matrix of $\left(\widehat{p}',\widehat{f}'\right)'$, the asymptotic covariance matrix for $\widehat{\theta}$ is $\left[\partial Z\left(\widehat{p},\widehat{f},\theta\right)/\partial\theta'\widehat{W}\partial Z\left(\widehat{p},\widehat{f},\theta\right)/\partial\theta\right]^{-1}$.
- When W is diagonal matrix, (2) is nonlinear least squares.
- When $\widetilde{u}_{jt}(x,\theta)$ is linear in θ, (2) has a closed form. Setting $W=\widehat{W}$:

$$\widehat{\theta}=\left\{\left[\partial Z\left(\widehat{p},\widehat{f},\theta\right)/\partial\theta\right]'\widehat{W}\left[\partial\partial Z\left(\widehat{p},\widehat{f},\theta\right)/\partial\theta\right]\right\}^{-1}\times\left[\partial Z\left(\widehat{p},\widehat{f},\theta\right)/\partial\theta\right]'\widehat{W}y\left(\widehat{p},\widehat{f}\right)$$

- Finally, the estimator carries over to the games case with minimal notational changes.

- How does finite dependence work when $\rho > 1$?
- Consider the following model of labor supply and human capital.
- In each of T periods an individual chooses whether to work, $d_{2t} = 1$, or stay home $d_{1t} = 1$. She acquires human capital, x_t, by working, with the payoff to working increasing in her human capital.
- If the individual works in period t, $x_{t+1} = x_t + 2$ with probability 0.5 and $x_{t+1} = x_t + 1$ also with probability 0.5.
- Every period after t, the human capital gain from working is fixed at one additional unit.
- When the individual does not work, her human capital remains the same in the next period.

Examples of Finite Dependence

Establishing finite dependence in the labor supply example

- First consider staying home at t and then work for the next two periods. Set:

$$\omega_{12t,t+1}(x_t, x_{t+1}) = \omega_{12t,t+2}(x_t, x_{t+2}) = 1$$

- This sequence of choices (stay home, work, work) increases human capital two units by $t+3$.
- Now consider working at t, staying home in period $t+2$, and depending on whether human capital increases by one or two units in t, work in $t+1$. Set:

$$\begin{aligned} \omega_{21t,t+1}(x_t, x_t+2) &= \omega_{21t,t+2}(x_t, x_t+2) = 1 \\ \omega_{22t,t+1}(x_t, x_t+1) &= \omega_{12t,t+2}(x_t, x_t+2) = 1 \end{aligned}$$

- These weights also increase the total the human capital stock by two units for sure.

Examples of Finite Dependence

An alternative way of establishing finite dependence in the labor supply example

- Consider working in period t and then staying home for the next two periods regardless of how much human capital is accumulated:

$$\begin{aligned} \omega_{21t,t+1}(x_t, x_t + 2) &= \omega_{21t,t+2}(x_t, x_t + 2) = 1 \\ \omega_{21t,t+1}(x_t, x_t + 1) &= \omega_{21t,t+2}(x_t, x_t + 1) = 1 \end{aligned}$$

- Now consider staying home in t, working in $t + 1$, and with probability one half working in period $t + 2$:

$$\begin{aligned} \omega_{12t,t+1}(x_t, x_t) &= 1 \\ \omega_{11t,t+2}(x_t, x_t + 1) &= \omega_{12t,t+2}(x_t, x_t + 1) = 1/2 \end{aligned}$$

- In both cases the exante distribution of human capital is the same:

$$\kappa_{1t2}\left(x_{t+3} \left| x_t \right.\right) = \kappa_{2t2}\left(x_{t+3} \left| x_t \right.\right) = \begin{cases} 1/2 & \text{if } x_{t+3} = x_t + 1 \\ 1/2 & \text{if } x_{t+3} = x_t + 2 \end{cases}$$

Examples of Finite Dependence

Nonstationary search model

- Consider a simple search model in which all jobs are temporary, last only one period.
- Each period $t \in \{1, \ldots, T\}$ an individual may stay home by setting $d_{1t} = 1$, or apply for temporary employment setting $d_{2t} = 1$.
- Job applicants are successful with probability λ_t
- The current utility from employment depends on experience, denoted by $x \in \{1, \ldots, X\}$.
- Experience increases by one unit with each period of work, and does not depreciate.
- The preference primitives are given by the current utility from staying home, denoted by $U_{1t}(x_t)$, and the utility from working, $U_{2t}(x_t)$.
- Thus the dynamics of the model come through experience.
- Nonstationarities arise through time varying offer arrival weights, λ_t, and wages (as indicated by t subscripts on current utilities).

Examples of Finite Dependence

Finite dependence in the nonstationary search model

- One period finite dependence is established by constructing two paths; one starts with staying home, $d_{1t} = 1$, the other begins with an employment application, $d_{2t} = 1$.
- Staying home is followed by applying for employment with weight $\lambda_t / \lambda_{t+1}$:

$$\omega_{12t,t+1}\left(x_t, x_t\right) = \lambda_t / \lambda_{t+1} = 1 - \omega_{11t,t+1}\left(x_t, x_t\right)$$

- Applying for employment is followed by staying home:

$$\omega_{21t,t+1}\left(x_t, x_t\right) = \omega_{21t,t+1}\left(x_t, x_t + 1\right) = 1$$

- Both sequences generate the same distribution for x_{t+2}:

$$\kappa_{1t1}(x_{t+2}|x_t) = \kappa_{2t1}(x_{t+2}|x_t) = \left\{ \begin{array}{l} 1 - \lambda_t \text{ for } x_{t+2} = x_t \\ \lambda_t \text{ for } x_{t+2} = x_t + 1 \end{array} \right.$$

- Notice that if $\lambda_t > \lambda_{t+1}$ then $\omega_{12t,t+1}\left(x_t, x_t\right) > 1$ and $\omega_{11t,t+1}\left(x_t, x_t\right) = 1 - \lambda_t / \lambda_{t+1} < 0$.

Determining whether Finite Dependence Exists

Intuition for establishing one period dependence in single agent settings

- One period finite dependence holds if there are weights such that:

$$\begin{aligned}\kappa_{it,t+1}(x_{t+2}|x_t) &\equiv \sum_x \sum_k \omega_{ikt,t+1}(x_t, x) f_{k,t+1}(x_{t+2}|x) f_{it}(x|x_t) \\ &= \sum_x \sum_k \omega_{jkt,t+1}(x_t, x) f_{k,t+1}(x_{t+2}|x) f_{jt}(x|x_t) \\ &\equiv \kappa_{jt,t+1}(x_{t+2}|x_{t+2})\end{aligned}$$

- Setting $\omega_{jJt,t+1} = 1 - \sum_{k=1}^{J-1} \omega_{jkt,t+1}$ we require for each x_{t+2}:

$$\begin{aligned}&\sum_x \sum_{k=1}^{J-1} [f_{it}(x|x_t) - f_{jt}(x|x_t)] f_{k,t+1}(x_{t+2}|x) \\ = &\sum_x \sum_{k=1}^{J-1} \omega_{ikt,t+1}(x_t, x) [f_{k,t+1}(x_{t+2}|x) - f_{J,t+1}(x_{t+2}|x)] f_{it}(x|x_t) \\ &- \sum_x \sum_{k=1}^{J-1} \omega_{jkt,t+1}(x_t, x) [f_{k,t+1}(x_{t+2}|x) - f_{J,t+1}(x_{t+2}|x)] f_{jt}(x|x_t)\end{aligned}$$

- This is a linear system to be solved in the $\omega_{ikt,t+1}(x_t, x)$ and $\omega_{jkt,t+1}(x_t, x)$ terms for each x_{t+2} using linear algebra.
- Nominally there are:
  1. $X - 1$ equations corresponding to the states in $t + 2$, since the remaining equation would be automatically satisfied.
  2. $2(J - 1)X$ weights, for i and j, for the first $J - 1$ choices, and each of the X states that the initial choice might lead to.
- Nevertheless the system has some special features which complicate matters. Some states might not be attainable in period:
  1. $t + 1$ from x_t for a given choice i and/or j. That reduces the number of ω weights to choose.
  2. $t + 2$ from for a given choice i and/or j, regardless of how the weights are chosen.
- The system is identified if the determinant of the ω coefficient vector for the matrix representation of these equations is nonzero.

Determining whether Finite Dependence Exists

How do we check for finite dependence beyond one period?

- Dynamic games hardly ever support one period dependence:
  - Intuitively different actions by one firm induce the other firms to take different equilibrium actions the next period.
  - Consequently the state space for the game players is mostly like to have a different distribution two periods later
- Many single agent problems also have finite dependence of more than one period (such as housing paper presented at the conference).
- If ρ period dependence fails, we check for $\rho + 1$ period dependence.
- Roughly speaking, to check for ρ period dependence we find the attainable states for period $t + \rho - 1$. Note this is not a combinatorial problem; we do it state-by-state.
- Then we form different combinations of states.
- For each combination we essentially go through the same exercise as for the one period dependence case.