Conditional Independence and the Inversion Theorem

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Recapitulation

A dynamic discrete choice model

- Each period $t \in \{1, 2, \ldots, T\}$ for $T \leq \infty$, an individual chooses among J mutually exclusive actions.
- Let d_{jt} equal one if action $j \in \{1, \ldots, J\}$ is taken at time t and zero otherwise:

$$d_{jt} \in \{0, 1\}$$

$$\sum_{j=1}^{J} d_{jt} = 1$$

- Suppose that actions taken at time t can potentially depend on the state $z_t \in Z$.
- The current period payoff at time t from taking action j is $u_{jt}^*(z_t)$.
- Given choices $(d_{1t}, \ldots, d_{Jt})$ in each period $t \in \{1, 2, \ldots, T\}$ the individual's expected utility is:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1} d_{jt} u_{jt}^*(z_t) \,|z_1\right\}$$

Recapitulation

Value function and optimization

- Write the optimal decision rule as $d_t^o(z_t) \equiv (d_{1t}^o(z_t), \ldots, d_{Jt}^o(z_t))$.
- Denote the value function by $V_t^*(z_t)$:

$$\begin{aligned} V_t^*(z_t) &\equiv E\left\{\sum_{s=t}^{T}\sum_{j=1}^{J}\beta^{t-1} d_{js}^o(z_s)\, u_{js}^*(z_s)\,|z_t\right\} \\ &= \sum_{j=1}^{J} d_{jt}^o\left[u_{jt}^*(z_t) + \beta\int_{z_{t+1}} V_{t+1}^*(z_{t+1})\, dF_{jt}(z_{t+1}\,|z_t)\right] \end{aligned}$$

- Let $v_{jt}^*(z_t)$ denote the flow payoff of action j plus the expected future utility of behaving optimally from period $t+1$ on:

$$v_{jt}^*(z_t) \equiv u_{jt}^*(z_t) + \beta\sum_{z_{t+1}=1}^{Z} V_{t+1}^*(z_{t+1})\, dF_{jt}(z_{t+1}\,|z_t)$$

- Bellman's principle implies:

$$d_{jt}^o(z_t) \equiv \prod_{k=1}^{K} I\left\{v_{jt}^*(z_t) \geq v_{kt}^*(z_t)\right\}$$

Recapitulation

Reformulating the primitives

- Partition the states $z_t \equiv (x_t, \epsilon_t)$ into:
  - those which are observed, x_t
  - and those that are unobserved, ϵ_t.
- Without loss of generality we can express $u^*_{jt}(z_t)$ as the sum of its conditional expectation on the observed variables plus a residual:

$$u^*_{jt}(x_t, \epsilon_t) \equiv E\left[u^*_{jt}(x_t, \epsilon_t) \left| x_t \right.\right] + \epsilon_{jt} \equiv u_{jt}(x_t) + \epsilon_{jt}$$

- For identification and estimation purposes we typically treat β, $u_{jt}(z_t)$, $dF_{jt}\left(z_{t+1} \middle| z_t\right)$ and $dG\left(\epsilon_1 \left| x_1 \right.\right)$, the density/probability for ϵ_1, as the primititves to our model.
- We often index the family of models we are considering (and limiting our search to), by say Θ.

Recapitulation

ML estimation

- The maximum likelihood (ML) estimator, $\theta_{ML} \in \Theta$ selects θ to maximize the joint probability (density) of the observed occurrences:

$$\prod_{n=1}^{N} \int_{\epsilon_T} \cdots \int_{\epsilon_1} \left[\begin{array}{l} \sum_{j=1}^{J} I\left\{d_{njT} = 1\right\} d_{jT}^{o}\left(x_{nT}, \epsilon_T\right) \times \\ \prod_{t=1}^{T-1} H_{nt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t \right.\right) dG\left(\epsilon_1 \left| x_{n1} \right.\right) \end{array} \right]$$

where:

$$\begin{array}{l} H_{nt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t \right.\right) \equiv \\ \sum_{j=1}^{J} I\left\{d_{njt} = 1\right\} d_{jt}^{o}\left(x_{nt}, \epsilon_t\right) dF_{jt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t \right.\right) \end{array}$$

is the probability (density) of the pair $\left(x_{n,t+1}, \epsilon_{t+1}\right)$ conditional on $\left(x_{nt}, \epsilon_t\right)$ when the observed choices are optimal for $\theta \in \Theta$.

Recapitulation

A computational challenge

- What are the computational challenges to large state space?
  1. Computing the value function;
  2. Solving for equilibrium in a multiplayer setting;
  3. Integrating over unobserved heterogeneity.
- These challenges suggest on several dimensions:
  1. Keep the dimension of the state space small;
  2. Assume all choices and outcomes are observed;
  3. Model unobserved states as a matter of computational convenience;
  4. Consider only one side of market to finesse equilibrium issues;
  5. Adopt parameterizations based on convenient functional forms.

Separable Transitions in the Observed Variables

A simplification

- Suppose the transition of the observed variables does not depend on the unobserved variables for all (j, t, x_t, ϵ_t):

$$F_{jt}\left(x_{t+1}\left|x_t, \epsilon_t\right.\right) = F_{jt}\left(x_{t+1}\left|x_t\right.\right)$$

- Assuming x_{t+1} conveys all the information of x_t for the purposes of forming probability distributions at $t+1$:

$$\begin{aligned} F_{jt}\left(x_{t+1}, \epsilon_{t+1}\left|x_t, \epsilon_t\right.\right) &\equiv G_{j,t+1}\left(\epsilon_{t+1}\left|x_{t+1}, x_t, \epsilon_t\right.\right) F_{jt}\left(x_{t+1}\left|x_t, \epsilon_t\right.\right) \\ &\equiv G_{j,t+1}\left(\epsilon_{t+1}\left|x_{t+1}, \epsilon_t\right.\right) F_{jt}\left(x_{t+1}\left|x_t\right.\right) \end{aligned}$$

- The ML estimator maximizes the same criterion function but $H_{nt}\left(x_{n,t+1}, \epsilon_{t+1}\left|x_{nt}, \epsilon_t\right.\right)$ simplifies to:

$$H_{nt}\left(x_{n,t+1}, \epsilon_{t+1}\left|x_{nt}, \epsilon_t\right.\right) =$$
$$\sum_{j=1}^{J} I\left\{d_{njt} = 1\right\} d_{jt}^{o}\left(x_{nt}, \epsilon_t\right) dG_{j,t+1}\left(\epsilon_{t+1}\left|x_{n,t+1}, \epsilon_t\right.\right) dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right.$$

Separable Transitions in the Observed Variables

Exploiting separability in estimation

- Instead of estimating all the parameters at once, we could use a two stage estimator to reduce computation costs:
  1. Estimate $F_{jt}\left(x_{t+1}\left|x_t\right.\right)$ with a cell estimator (for x finite), a nonparametric estimator, or a parametric function;
  2. Define:

$$\widehat{H}_{nt}\left(x_{n,t+1}, \epsilon_{t+1}\left|x_{nt}, \epsilon_t\right.\right) \equiv \sum_{j=1}^{J}\left[\begin{array}{l} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt}, \epsilon_t\right) \\ \times dG_{j,t+1}\left(\epsilon_{t+1}\left|x_{n,t+1}, \epsilon_t ; \theta\right.\right) d\widehat{F}_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right) \end{array}\right]$$

  3. Select the remaining (preference) parameters to maximize:

$$\prod_{n=1}^{N} \int_{\epsilon_T} \cdots \int_{\epsilon_1}\left[\begin{array}{l} \sum_{j=1}^{J} I\left\{d_{njT}=1\right\} d_{jT}^{o}\left(x_{nT}, \epsilon_T\right) \times \\ \prod_{t=1}^{T-1} \widehat{H}_{nt}\left(x_{n,t+1}, \epsilon_{t+1}\left|x_{nt}, \epsilon_t\right.\right) dG_1\left(\epsilon_1\left|x_{n1}\right.\right) \end{array}\right]$$

  4. Correct standard errors from the first stage estimator to account for the loss in asymptotic efficiency.

- Separable transitions do not, however, free us from:
  1. the curse of multiple integration;
  2. numerical optimization to obtain the value function.
- Suppose in addition, that conditional on x_{t+1}, the unobserved variable ϵ_{t+1} is independent of (x_t, ϵ_t, d_t).
- Conditional independence embodies both assumptions:

$$\begin{aligned} dF_{jt}\left(x_{t+1} \left| x_t, \epsilon_t \right.\right) &= dF_{jt}\left(x_{t+1} \left| x_t \right.\right) \\ dG_{j,t+1}\left(\epsilon_{t+1} \left| x_{t+1}, x_t, \epsilon_t \right.\right) &= dG_{t+1}\left(\epsilon_{t+1} \left| x_{t+1} \right.\right) \end{aligned}$$

- It implies:

$$dF_{jt}\left(x_{t+1}, \epsilon_{t+1} \left| x_t, \epsilon_t \right.\right) = dF_{jt}\left(x_{t+1} \left| x_t \right.\right) dG_{t+1}\left(\epsilon_{t+1} \left| x_{t+1} \right.\right)$$

Conditional Independence

Exante value functions and conditional value functions defined

- Given conditional independence, define the exante valuation function as:

$$V_t(x_t) \equiv E\left[V_t^*(x_t, \epsilon_t)\left|x_t\right.\right]$$

and the conditional valuation function as:

$$v_{jt}(x_t) \equiv u_{jt}(x_t) + \beta \int_{x_{t+1}} V_{t+1}(x_{t+1}) dF_{jt}\left(x_{t+1}\left|x_t\right.\right)$$

- Optimal behavior implies that $d_{jt}^o(x_t, \epsilon) = 1$ if and only if:

$$\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x_t) - v_{kt}(x_t)$$

for all $k \in \{1, \ldots, J\}$.

Conditional Independence

Conditional choice probabilities defined

- Under conditional independence, the conditional choice probability (CCP) for action j is defined for each (t, x_t, j) as the probability of observing the j^{th} choice conditional on the values of the observed variables when behavior is optimal:

$$p_{jt}(x_t) \equiv E\left[d_{jt}^o(x_t, \epsilon_t)\,|x_t\right] = \int_{\epsilon_t} d_{jt}^o(x_{nt}, \epsilon_t)\, g_t(\epsilon_t\,|x_{nt})\, d\epsilon_t$$

  where we now assume (following the literature) that $G_t(\epsilon_t\,|x_{nt})$ has probability density function $g_t(\epsilon_t\,|x_{nt})$.
- The previous slide now implies:

$$p_{jt}(x_t) = \int_{\epsilon_t} \prod_{k=1}^{J} I\left\{\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x_{nt}) - v_{kt}(x_{nt})\right\} g_t(\epsilon_t\,|x_t)\, d\epsilon_t$$

Conditional Independence

Simplifying expressions within the likelihood

- Conditional independence simplifies $H_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right)$ to:

$$H_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right)=$$
$$\sum_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right) g_{t+1}\left(\epsilon_{t+1}\left|x_{n,t+1}\right.\right) dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)$$

- Also note that:

$$\begin{aligned} & \prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right) dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)\right\} \\ = & \prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)\right\} \\ & \times\prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right)\right\} \end{aligned}$$

- Hence the contribution of $n \in \{1, \ldots, N\}$ to the likelihood is the product of:

$$\prod_{t=1}^{T-1} \sum_{j=1}^{J} I \{d_{njt} = 1\} dF_{jt} (x_{n,t+1} | x_{nt})$$

and:

$$\int_{\epsilon_T} \cdots \int_{\epsilon_1} \prod_{t=1}^{T-1} \sum_{j=1}^{J} \left[\begin{array}{l} I \{d_{njt} = 1\} d_{jt}^{o} (x_{nt}, \epsilon_t) \\ \times g_{t+1} (\epsilon_{t+1} | x_{n,t+1}) g_1 (\epsilon_1 | x_{n1}) d\epsilon_1 \ldots d\epsilon_T \end{array} \right]$$
$$= \prod_{t=1}^{T} \left[\sum_{j=1}^{J} I \{d_{njt} = 1\} \int_{\epsilon_t} d_{jt}^{o} (x_{nt}, \epsilon_t) g_t (\epsilon_t | x_{nt}) d\epsilon_t \right]$$

- Since:

$$p_{jt}(x_t) \equiv \int_{\epsilon_t} d_{jt}^o(x_{nt}, \epsilon_t)\, g_t(\epsilon_t | x_{nt})\, d\epsilon_t = E\left[d_{jt}^o(x_t, \epsilon_t) | x_t\right]$$

the log likelihood can now be compactly expressed as:

$$\sum_{n=1}^{N} \sum_{t=1}^{T-1} \sum_{j=1}^{J} I\{d_{njt} = 1\} \ln\left[dF_{jt}(x_{n,t+1} | x_{nt})\right]$$
$$+ \sum_{n=1}^{N} \sum_{t=1}^{T} \sum_{j=1}^{J} I\{d_{njt} = 1\} \ln p_{jt}(x_t)$$

- Suppose we only had data on the last period T, and wished to estimate the preferences determining choices in T.
- By definition this is a static problem in which $v_{jT}(x_T) \equiv u_{jT}(x_T)$.
- For example to the probability of observing the J^{th} choice is:

$$p_{JT}(x_T) \equiv \int_{-\infty}^{\epsilon_{JT}+u_{JT}(x_T)-u_{1T}(x_T)} \ldots \int_{-\infty}^{\epsilon_{JT}+u_{JT}(x_T)-u_{J-1,T}(x_T)} \int_{-\infty}^{\infty} g_T(\epsilon_T | x_T) \, d\epsilon_T$$

- The only essential difference between a estimating a static discrete choice model using ML and a estimating a dynamic model satisfying conditional independence using ML is that parametrizations of $v_{jt}(x_t)$ based on $u_{jt}(x_t)$ do not have a closed form, but must be computed numerically.

- The starting point for our analysis is to define differences in the conditional valuation functions as:

$$\Delta v_{jkt}(x) \equiv v_{jt}(x) - v_{kt}(x)$$

- Although there are $J(J-1)$ differences all but $(J-1)$ are linear combinations of the $(J-1)$ basis functions.
- For example setting the basis functions as:

$$\Delta v_{jt}(x) \equiv v_{jt}(x) - v_{Jt}(x)$$

then clearly:

$$\Delta v_{jkt}(x) = \Delta v_{jt}(x) - \Delta v_{kt}(x)$$

- Without loss of generality we focus on this particular basis function.

Inversion

Each CCP is a mapping of differences in the conditional valuation functions

- Using the definition of $\Delta v_{jt}(x)$:

$$
\begin{aligned}
p_{jt}(x) &\equiv \int d_{jt}^{o}(x,\epsilon)\, g_t(\epsilon \,|x)\, d\epsilon \\
&= \int I\{\epsilon_k \leq \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall\ k \neq j\}\, g_t(\epsilon \,|x)\, d\epsilon \\
&= \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{1t}(x)} \cdots \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{J-1,t}(x)} \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)} g_t(\epsilon \,|x)\, d\epsilon
\end{aligned}
$$

- Noting $g_t(\epsilon \,|x) \equiv \partial^J G_t(\epsilon \,|x) / \partial\epsilon_1, \ldots, \partial\epsilon_J$, integrate over $(\epsilon_1, \ldots, , \epsilon_{j-1}, \epsilon_{j+1} \ldots, \epsilon_J)$.
- Denoting $G_{jt}(\epsilon \,|x) \equiv \partial G_t(\epsilon \,|x) / \partial\epsilon_j$, yields:

$$
p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \ldots \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} \,|x \right) d\epsilon_j
$$

Inversion

There are as many CCPs as there are conditional valuation functions

- For any vector $J-1$ dimensional vector $\delta \equiv (\delta_1, \ldots, \delta_{J-1})$ define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}\left(\epsilon_j + \delta_j - \delta_1, \ldots, \epsilon_j, \ldots, \epsilon_j + \delta_j \,|x\right) d\epsilon_j$$

- We interpret $Q_{jt}(\delta, x)$ as the probability taking action j in a static random utility model (RUM) where the payoffs are $\delta_j + \epsilon_j$ and the probability distribution of disturbances is given by $G_t(\epsilon\,|x)$.
- It follows from the definition of $Q_{jt}(\delta, x)$ that:

$$0 \leq Q_{jt}(\delta, x) \leq 1 \text{ for all } (j, t, \delta, x) \text{ and } \sum_{j=1}^{J-1} Q_{jt}(\delta, x) \leq 1$$

- In particular the previous slide implies that for any given (j, t, x):

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x\right) d\epsilon_j \equiv Q_{jt}(\Delta v_t(x), x)$$

Theorem (Inversion)

For each (t, δ, x) *define:*

$$Q_t(\delta, x) \equiv \left(Q_{1t}(\delta, x), \ldots Q_{J-1,t}(\delta, x)\right)'$$

Then the vector function $Q_t(\delta, x)$ *is invertible in* δ *for each* (t, x).

- Note that $p_{Jt}(x) = Q_{Jt}(\Delta v_t, x)$ is a linear combination of the other equations in the system because $\sum_{k=1}^{J} p_k = 1$.
- Let $p \equiv (p_1, \ldots, p_{J-1})$ where $0 \leq p_j \leq 1$ for all $j \in \{1, \ldots, J-1\}$ and $\sum_{j=1}^{J-1} p_j \leq 1$. Denote the inverse of $Q_{jt}(\Delta v_t, x)$ by $Q_{jt}^{-1}(p, x)$.
- The inversion theorem implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}\left[p_t(x), x\right] \\ \vdots \\ Q_{J-1,t}^{-1}\left[p_t(x), x\right] \end{bmatrix}$$

- In what sense does the inversion theorem help us to finesse optimization and integration by exploiting conditional independence?
- We use the Inversion Theorem to:
    1. provide empirically tractable representations of the conditional value functions.
    2. analyze identification in dynamic discrete choice models.
    3. provide convenient parametric forms for the density of ϵ_t that generalize the Type 1 Extreme Value distribution.
    4. provide cheap estimators for dynamic discrete choice models and dynamic discrete choice games of incomplete information.
    5. introduce new methods for incorporating unobserved state variables.

Corollaries of the Inversion Theorem

Identifying the policy function

- From the definition of the optimal decision rule, and then appealing to the inversion theorem:

$$
\begin{aligned}
d_{jt}^{o}(x_t, \epsilon_t) &= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x) - v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq \begin{array}{l} v_{jt}(x) - v_{Jt}(x_t) \\ - \left[v_{kt}(x) - v_{Jt}(x_t)\right] \end{array}\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq \Delta v_{jt}(x) - \Delta v_{kt}(x)\right\} \\
&= \prod\nolimits_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq Q_{jt}^{-1}\left[p_t(x), x\right] - Q_{kt}^{-1}\left[p_t(x), x\right]\right\}
\end{aligned}
$$

- If $G_t(\epsilon \,|x)$ is known and the data generating process (DGP$)$ is (x_t, d_t), then $p_t(x)$ and hence $d_t^{o}(x_t, \epsilon_t)$ are identified.

- Define the conditional value function correction as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- In stationary settings, we drop the t subscript and write:

$$\psi_j(x) \equiv V(x) - v_j(x)$$

- Suppose that instead of taking the optimal action she committed to taking action j instead. Then the expected lifetime utility would be:

$$v_{jt}(x_t) + E_t[\epsilon_{jt} | x_t]$$

so committing to j before ϵ_t is revealed entails a loss of:

$$V_t(x_t) - v_{jt}(x_t) - E_t[\epsilon_{jt} | x_t] = \psi_{jt}(x) - E_t[\epsilon_{jt} | x_t]$$

- For example if $E_t[\epsilon_t | x_t] = 0$, the loss simplifies to $\psi_{jt}(x)$.

Corollaries of the Inversion Theorem

Identifying the conditional value function correction

- From their respective definitions:

$$
\begin{aligned}
& V_t(x) - v_{it}(x) \\
= & \sum_{j=1}^{J} \left\{ p_{jt}(x) \left[v_{jt}(x) - v_{it}(x) \right] + \int \epsilon_{jt} d_{jt}^{o} \left(x_t, \epsilon_t \right) g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t \right\}
\end{aligned}
$$

- But:

$$
v_{jt}(x) - v_{it}(x) = Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{it}^{-1} \left[p_t(x), x \right]
$$

and

$$
\begin{aligned}
& \int \epsilon_{jt} d_{jt}^{o} \left(x, \epsilon_t \right) g \left(\epsilon_t \left| x \right. \right) d\epsilon_t \\
= & \int \prod_{k=1}^{J} 1 \left\{ \begin{array}{l} \epsilon_{kt} - \epsilon_{jt} \\ \leq Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{kt}^{-1} \left[p_t(x), x \right] \end{array} \right\} \epsilon_{jt} g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t
\end{aligned}
$$

- Therefore $\psi_{it}(x) \equiv V_t(x) - v_{it}(x)$ is identified if $G_t(\epsilon \left| x \right.)$ is known.

Conditional Valuation Function Representation

Telescoping one period forward

- From its definition:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}(x_{t+1}|x_t)$$

- Substituting for $V_{t+1}(x_{t+1})$ using conditional value function correction we obtain for any k:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} \left[v_{k,t+1}(x) + \psi_{k,t+1}(x) \right] f_{jt}(x|x_t)$$

- We could repeat this procedure ad infinitum, substituting in for $v_{k,t+1}(x)$ by using the definition for $\psi_{kt}(x)$.

Conditional Valuation Function Representation

Recursively defining the distribution of future state variables

- To formalize this idea, consider a random sequence of weights from t to T which begins with $\omega_{jt}(x_t, j) = 1$.
- For periods $\tau \in \{t+1, \ldots, T\}$, the choice sequence maps x_τ and the initial choice j into

$$\omega_\tau(x_\tau, j) \equiv \{\omega_{1\tau}(x_\tau, j), \ldots, \omega_{J\tau}(x_\tau, j)\}$$

where $\omega_{k\tau}(x_\tau, j)$ may be negative or exceed one but:

$$\sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) = 1$$

.

- The weight of state $x_{\tau+1}$ conditional on following the choices in the sequence is recursively defined by $\kappa_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and for $\tau = t+1, \ldots, T$:

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) f_{k\tau}(x_{\tau+1}|x_\tau)\kappa_{\tau-1}(x_\tau|x_t, j)$$

Theorem (Representation)

For any state $x_t \in \{1, \ldots, X\}$, choice $j \in \{1, \ldots, J\}$ and weights $\omega_\tau(x_\tau, j)$ defined for periods $\tau \in \{t, \ldots, T\}$:

$$
\begin{aligned}
v_{jt}(x_t) &= u_{jt}(x_t) \\
&\quad + \sum_{\tau=t+1}^{T} \sum_{k=1}^{J} \sum_{x=1}^{X} \beta^{\tau-t} \left[u_{k\tau}(x) + \psi_k [p_\tau(x)] \right] \omega_{k\tau}(x, j) \kappa_{\tau-1}(x | x_t, j
\end{aligned}
$$

- The theorem yields an alternative expression for $v_{jt}(x_t)$ that dispenses with recursive maximization.
- Intuitively, the individuals have already solved their optimization problem, so their decisions, as reflected in their CCPs, are informative of their value functions.

- Can we exploit this representation in identification and estimation?
- To make the approach operational requres us to compute $\psi_k(p)$ for at least some k.
- Suppose ϵ is drawn from the GEV distribution function:

$$G(\epsilon_1, \epsilon_2, \ldots, \epsilon_J) \equiv \exp\left[-\mathcal{H}\left(\exp[-\epsilon_1], \exp[-\epsilon_2], \ldots, \exp[-\epsilon_J]\right)\right]$$

where $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ satisfies the following properties:

1. $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ is nonnegative, real valued, and homogeneous of degree one;
2. $\lim \mathcal{H}(Y_1, Y_2, \ldots, Y_J) \to \infty$ as $Y_j \to \infty$ for all $j \in \{1, \ldots, J\}$;
3. for any distinct $(i_1, i_2, \ldots, i_r)$ the cross derivative $\partial\mathcal{H}(Y_1, Y_2, \ldots, Y_J) / \partial Y_{i_1}, Y_{i_2}, \ldots, Y_{i_r}$ is nonnegative for r odd and nonpositive for r even.

- Suppose $G(\epsilon)$ factors into two independent distributions, one a nested logit, and the other any GEV distribution.
- Let $\mathcal{J}$ denote the set of choices in the nest and denote the other distribution by $G_0(Y_1, Y_2, \ldots, Y_K)$ let K denote the number of choices that are outside the nest.
- Then:

$$G(\epsilon) \equiv G_0(\epsilon_1, \ldots, \epsilon_K) \exp\left[-\left(\sum_{j \in \mathcal{J}} \exp\left[-\epsilon_j/\sigma\right]\right)^{\sigma}\right]$$

- The correlation of the errors within the nest is given by $\sigma \in [0,1]$ and errors within the nest are uncorrelated with errors outside the nest. When $\sigma = 1$, the errors are uncorrelated within the nest, and when $\sigma = 0$ they are perfectly correlated.

- Define $\phi_j(Y)$ as a mapping into the unit interval where

$$\phi_j(Y) = Y_j \mathcal{H}_j \left(Y_1, \ldots, Y_J\right) / \mathcal{H}\left(Y_1, \ldots, Y_J\right)$$

- Since $\mathcal{H}_j\left(Y_1, \ldots, Y_J\right)$ and $\mathcal{H}\left(Y_1, \ldots, Y_J\right)$ are homogeneous of degree zero and one respectively, $\phi_j(Y)$ is a probability, because $\phi_j(Y) \geq 0$ and $\sum_{j=1}^{J} \phi_j(Y) = 1$.

Lemma (GEV correction factor)

When ϵ_t is drawn from a GEV distribution, the inverse function of $\phi(Y) \equiv (\phi_2(Y), \ldots \phi_J(Y))$ exists, which we now denote by $\phi^{-1}(p)$, and:

$$\psi_j(p) = \ln \mathcal{H}\left[1, \phi_2^{-1}(p), \ldots, \phi_J^{-1}(p)\right] - \ln \phi_j^{-1}(p) + \gamma$$

Lemma

For the nested logit $G(\epsilon_t)$ defined above:

$$\psi_j(p) = \gamma - \sigma \ln(p_j) - (1-\sigma) \ln\left(\sum_{k \in \mathcal{J}} p_k\right)$$

- Note that $\psi_j(p)$ only depends on the conditional choice probabilities for choices that are in the nest: the expression is the same no matter how many choices are outside the nest or how those choices are correlated.
- Hence, $\psi_j(p)$ will only depend on $p_{j'}$ if ϵ_{jt} and $\epsilon_{j't}$ are correlated. When $\sigma = 1$, ϵ_{jt} is independent of all other errors and $\psi_j(p)$ only depends on p_j.

Adapting Dynamic Games to the CCP Framework

Players and choices

- This framework naturally lends itself to studying equilibrium in games of incompete information.
- For example consider a dynamic infinite horizon game for finite I players.
- Thus $T = \infty$ and $I < \infty$.
- Each player $i \in I$ makes a choice $d_t^{(i)} \equiv \left(d_{1t}^{(i)}, \ldots, d_{Jt}^{(i)} \right)$ in period t.
- Denote the choices of all the players in period t by:

$$d_t \equiv \left(d_t^{(1)}, \ldots, d_t^{(I)} \right)$$

and denote by:

$$d_t^{(-i)} \equiv \left(d_t^{(1)}, \ldots, d_t^{(i-1)}, d_t^{(i+1)}, \ldots, d_t^{(I)} \right)$$

the choices of $\{1, \ldots, i-1, i+1, \ldots, I\}$ in period t, that is all the players apart from i.

- Denote by x_t the state variables of the game that are not *iid*.
- For example x_t includes the capital of every firm. Then:
  - firms would have the same state variables.
  - x_t would affect rivals in very different ways.
- We assume all the players observe x_t.
- Denote by $F\left(x_{t+1}\left|x_t, d_t\right.\right)$ the probability of x_{t+1} occurs when the state variables are x_t and the players collectively choose d_t.
- Similarly let:

$$F_j\left(x_{t+1}\left|x_t, d_t^{(-i)}\right.\right) \equiv F\left(x_{t+1}\left|x_t, d_t^{(-i)}, d_{jt}^{(i)} = 1\right.\right)$$

denote the probability distribution determining x_{t+1} given x_t when $\{1, \ldots, i-1, i+1, \ldots, I\}$ choose $d_t^{(-i)}$ in t and i makes choice j.

- Suppose $\epsilon_t^{(i)} \equiv \left(\epsilon_{1t}^{(i)}, \ldots, \epsilon_{Jt}^{(i)}\right)$, identically and independently distributed with density $g\left(\epsilon_t^{(i)}\right)$, affects the payoffs of i in t.
- Also let $\epsilon_t^{(-i)} \equiv \left(\epsilon_t^{(1)}, \ldots, \epsilon_t^{(i-1)}, \epsilon_t^{(i+1)}, \ldots, \epsilon_t^{(I)}\right)$.
- The systematic component of current utility or payoff to player i in period t form taking choice j when everybody else chooses $d_t^{(-i)}$ and the state variables are z_t is denoted by $U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$.
- Denoting by $\beta \in (0, 1)$ the discount factor, the summed discounted payoff to player i throughout the course of the game is:

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} d_{jt}^{(i)} \left[U_j^{(i)}\left(x_t, d_t^{(-i)}\right) + \epsilon_{jt}^{(i)}\right]$$

- Players noncooperatively maximize their expected utilities, moving simultaneously each period. Thus i does not condition on $d_t^{(-i)}$ when making his choice at date t, but only sees $\left(x_t, \epsilon_t^{(i)}\right)$.

Adapting Dynamic Games to the CCP Framework

Markov strategies

- This is a stationary environment and we focus on Markov decision rules, which can be expressed $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$.
- Let $d^{(-i)}\left(x_t, \epsilon_t^{(-i)}\right)$ denote the strategy of every player but i:

$$\left(\begin{array}{r} d^{(1)}\left(x_t, \epsilon_t^{(1)}\right), \ldots, d^{(i-1)}\left(x_t, \epsilon_t^{(i-1)}\right), d^{(i+1)}\left(x_t, \epsilon_t^{(i+1)}\right), \\ d^{(i+2)}\left(x_t, \epsilon_t^{(i+2)}\right) \ldots, d^{(I)}\left(x_t, \epsilon_t^{(I)}\right) \end{array}\right)$$

- Then the expected value of the game to i from playing $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$ when everyone else plays $d\left(x_t, \epsilon_t^{(-i)}\right)$ is:

$$V^{(i)}(x_1) \equiv$$
$$E\left\{\sum_{t=1}^{\infty} \sum_{j=1}^{J} \beta^{t-1} d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)\left[U_j^{(i)}\left(z_t, d\left(x_t, \epsilon_t^{(-i)}\right)\right)+\epsilon_{jt}^{(i)}\right] | x_1\right\}$$

- Integrating over $\epsilon_t^{(i)}$ we obtain the j^{th} conditional choice probability for the i^{th} player at t as $p_j^{(i)}(x_t)$:

$$p_j^{(i)}(x_t) = \int d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right) g\left(\epsilon_t^{(i)}\right) d\epsilon_t^{(i)}$$

- Let $P\left(d_t^{(-i)} \left| x_t\right.\right)$ denote the joint probability firm i's competitors choose $d_t^{(-i)}$ conditional on the state variables z_t.
- Since $\epsilon_t^{(i)}$ is distributed independently across $i \in \{1, \ldots, I\}$:

$$P\left(d_t^{(-i)} \left| x_t\right.\right) = \prod_{\substack{i'=1 \\ i' \neq i}}^{I} \left(\sum_{j=1}^{J} d_{jt}^{(i')} p_j^{(i')}(x_t)\right)$$

- The strategy $\left\{ d^{(i)}\left(x_t, \epsilon_t^{(i)}\right)\right\}_{i=1}^{I}$ is a Markov perfect equilibrium if, for all $\left(i, x_t, \epsilon_t^{(i)}\right)$, the best response of i to $d^{(-i)}\left(x_t, \epsilon_t^{(-i)}\right)$ is $d^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$ when everybody uses the same strategy thereafter.
- That is, suppose the other players collectively use $d^{(-i)}\left(x_t, \epsilon_t^{(-i)}\right)$ in period t, and $V^{(i)}(x_{t+1})$ is formed from $\left\{ d^{(i)}\left(x_t, \epsilon_t^{(i)}\right)\right\}_{i=1}^{I}$.
- Then $d^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$ solves for i choosing j to maximize:

$$\sum_{d_t^{(-i)}} P\left(d_t^{(-i)} \middle| x_t\right) \left\{ \begin{array}{l} U_j^{(i)}\left(x_t, d_t^{(-i)}\right) \\ +\beta \sum_{z=1}^{X} V^{(i)}(x) F_j\left(x \middle| x_t, d_t^{(-i)}\right) \end{array} \right\} + \epsilon_{jt}^{(i)}$$

Adapting Dynamic Games to the CCP Framework

Connection to Individual Optimization

- In equilibrium, the systematic component of the current utility of player i in period t, as a function of x_t, the state variables for game, and his own decision j, is:

$$u_j^{(i)}(x_t) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} \left| x_t\right.\right) U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$$

- Similarly the probability transition from x_t to x_{t+1} given action j by firm i is given by:

$$f_j^{(i)}\left(x_{t+1} \left| x_t^{(i)}\right.\right) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} \left| x_t^{(i)}\right.\right) F_j\left(x_{t+1} \left| x_t, d_t^{(-i)}\right.\right)$$

- The setup for player i is now identical to the optimization problem described in the second lecture for a stationary environment.

- Both theorems apply to this multiagent setting with two critical differences, and both are relevant for studying identification:
  1. $u_{jt}(x_t)$ is a primitive in single agent optimization problems, but $u_{jt}^{(i)}(x_t)$ is a reduced form parameter found by integrating $U_{jt}^{(i)}\left(x_t, d_t^{(\sim i)}\right)$ over the joint probability distribution $P_t\left(d_t^{(\sim i)} \left| x_t\right.\right)$.
  2. $f_{jt}\left(x_{t+1} \left| x_t\right.\right)$ is a primitive in single agent optimization problems, but $f_{jt}^{(i)}\left(x_{t+1} \left| x_t\right.\right)$ depends on CCPs of the other players, $P_t\left(d_t^{(\sim i)} \left| x_t\right.\right)$, as well as the primitive $F_{jt}\left(x_{t+1} \left| x_t, d_t^{(\sim i)}\right.\right)$. It is easy to interpret restrictions placed directly on $f_{jt}\left(x_{t+1} \left| x_t\right.\right)$ but placing restrictions on $F_{jt}\left(x_{t+1} \left| x_t, d_t^{(\sim i)}\right.\right)$ complicates matters in dynamic games because of the endogenous effects arising from $P_t\left(d_t^{(\sim i)} \left| x_t\right.\right)$ on $f_{jt}^{(i)}\left(x_{t+1} \left| x_t\right.\right)$.